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Tuning non-Hermitian skin effect and topological phase transition by next-nearest-neighbor hopping in electrical circuits

Cheng-Jun Han¹ , Rui-Xue Tang¹, Xu-Wei Li¹, Mo-Nan Wang¹, Jin-Yan Hu¹, Jun-Bo Zhao¹,
Shao-Ze Wang¹, Yi-Ping Wang^{1,*}  and Ai-Xi Chen^{2,*} 

¹ College of Science, Northwest A & F University, Yangling 712100, People's Republic of China

² Department of Physics, Zhejiang Sci-Tech University, Hangzhou 310018, People's Republic of China

* Authors to whom any correspondence should be addressed.

E-mail: ypwang2019@nwfufu.edu.cn and aixichen@zstu.edu.cn

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Abstract

Non-Hermitian topological phases and the non-Hermitian skin effect (NHSE) have attracted significant attention in recent years, yet effectively controlling these phenomena remains challenging. Here, we propose a one-dimensional non-Hermitian Su–Schrieffer–Heeger model implemented on a circuit platform, incorporating a tunable next-nearest-neighbor (NNN) hopping element. We systematically investigate the influence of NNN hopping on both topological properties and the NHSE. By mapping the circuit Laplacian to the Hamiltonian, we analyze the effects of NNN hopping on the admittance spectrum, eigenfrequency spectrum, and spatial impedance distribution. Our results demonstrate that increasing the NNN hopping strength suppresses the NHSE, driving boundary-localized eigenstates to extend into the bulk, a transition quantitatively characterized by the inverse participation ratio. Further analysis using non-Bloch band theory reveals that while NNN hopping does not alter the quantized values of topological invariants, it shifts the critical parameters of topological phase transitions. This work provides a theoretical platform for simulating and controlling non-Hermitian topological states in circuits and establishes a quantitative foundation, through generalized Brillouin zone deformation and non-Bloch invariants, for designing novel non-Hermitian topological devices based on coupling engineering.

1. Introduction

Topological phases in condensed matter physics have attracted significant interest in recent years, particularly with the introduction of non-Hermitian properties in topological insulators, leading to novel phenomena such as the non-Hermitian skin effect (NHSE) and exceptional points [1–7]. Non-Hermitian systems exhibit more complex spectral structures and richer topological features than their Hermitian counterparts, with the NHSE [8–15] and exceptional points [16–20] standing out as distinctive hallmarks. Under open boundary conditions (OBCs), eigenstates in non-Hermitian systems localize exponentially near the boundary a signature of the skin effect while exceptional points arise when eigenvalues and eigenvectors coalesce at specific parameter values. In Hermitian systems, topological properties are typically characterized by invariants defined on the Brillouin zone. In non-Hermitian systems, however, the conventional bulk-edge correspondence [21–23] often breaks down, rendering topological invariants defined under periodic boundary conditions (PBCs) inadequate for predicting boundary states under open boundaries. To overcome this limitation, non-Bloch band theory has been developed [24–27]. This framework extends the Brillouin zone to a generalized Brillouin zone (GBZ) [28] and introduces non-Bloch topological invariants [29–32], providing a rigorous theoretical description of non-Hermitian topology [33–39].

While significant progress has been made in realizing topological insulators in optical, acoustic, and mechanical, and cavity magnonic systems [40–43], their fabrication often involves complex processes and stringent experimental conditions. Unlike photon-mediated interactions in waveguide QED, the

next-nearest-neighbor (NNN) hopping in our lumped-element electrical circuit is directly engineered and independently tunable via circuit parameters. This feature makes the circuit platform particularly suitable for implementing the present model. Realizing non-Hermitian topological phases in quantum [44–49] and photonic platforms [50–55] remains challenging, primarily due to the difficulty in implementing controllable gain and loss. In this context, electrical circuit systems have emerged as a promising platform due to their high parametric stability, strong tunability, and low sensitivity to environmental perturbations. Moreover, gain and loss can be conveniently implemented using active components such as operational amplifiers, enabling the observation of rich non-Hermitian phenomena, including parity-time (PT)-symmetric phase transitions [56–58] and non-reciprocal transmission within a single, compact setup. Thus, circuit systems provide a versatile and promising testbed for exploring topological physics in both Hermitian and non-Hermitian regimes, with potential for future experimental realizations. Despite considerable progress, the effective dynamic control of the NHSE and the modulation of topological phases in non-Hermitian systems remain key challenges. It is also worth noting that most existing studies, including the present work, have been conducted theoretically and numerically; direct experimental demonstrations in this direction are still relatively limited. Notably, previous studies on circuit platforms have largely focused on non-Hermitian models with nearest-neighbor couplings [51–67]. However, the coupling range a fundamental and tunable degree of freedom in lattice models and its systematic role in regulating non-Hermitian systems, affecting topological phase transitions, the skin effect, and the bulk-boundary correspondence, have remained underexplored in experimentally feasible platforms. Its underlying physical mechanisms and potential applications await further clarification.

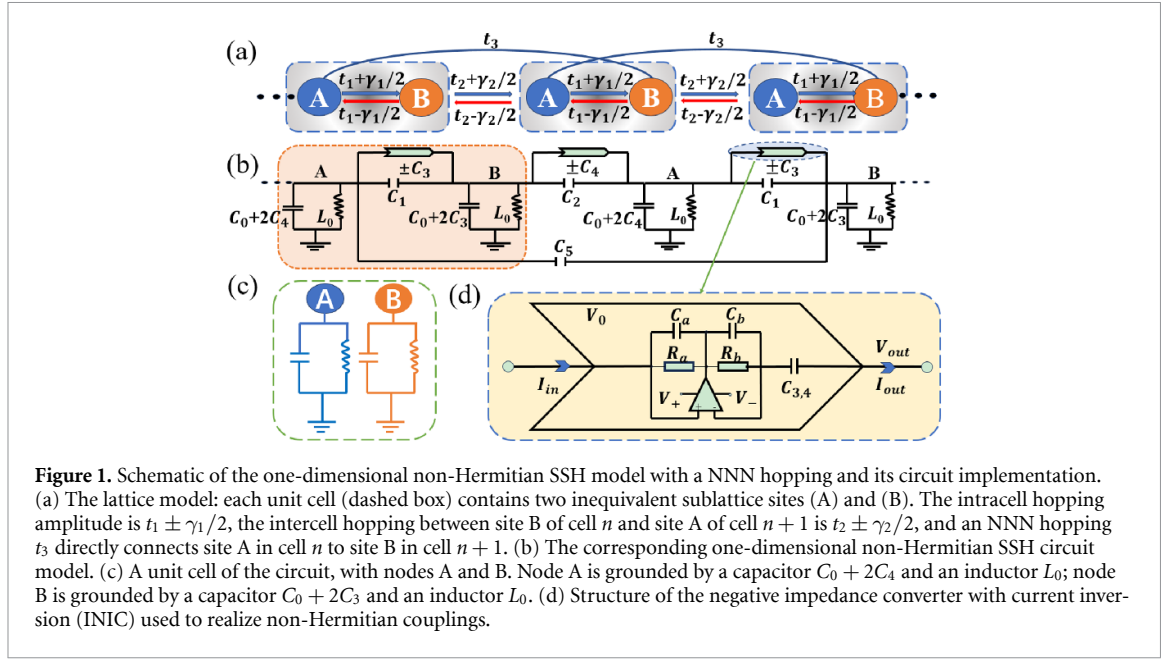
In this work, we investigate the influence of NNN hopping on the topological properties of a non-Hermitian Su–Schrieffer–Heeger (SSH) model. We first derive the real-space Hamiltonian with NNN hopping and obtain its Bloch form. The model is then mapped onto an electrical circuit, where the grounded Laplacian corresponds directly to the Hamiltonian. By varying the NNN hopping strength, we systematically analyze its impact on the admittance spectrum, eigenfrequency spectrum, spatial impedance profiles, and eigenstate distributions, with a focus on the NHSE. The inverse participation ratio is used to quantify the transition from localized to extended states. Furthermore, within the framework of non-Bloch band theory, we explore how NNN hopping alters the GBZ and the associated non-Bloch topological invariants, revealing how coupling-range engineering reshapes non-Hermitian topological phases. Our results show that increasing the NNN hopping suppresses the NHSE, drives the GBZ toward the unit circle, and induces a discrete jump of the non-Bloch winding number, signaling a topological phase transition. This work establishes NNN hopping as a controllable parameter for manipulating non-Hermitian topological states and provides a theoretical foundation for designing tunable topological devices.

This paper is organized as follows. In section 2, we derive the real-space and momentum-space Hamiltonians and establish their relation to the circuit Laplacian. In sections 3–5, we discuss the evolution of topological properties under NNN hopping and present circuit simulations to validate the theoretical results. Finally, section 6 summarizes our conclusions.

2. Model and circuit implementation

We consider a one-dimensional non-Hermitian SSH model with NNN hopping, as depicted in figure 1(a). Each unit cell consists of two inequivalent sublattice sites, labeled A and B. In real space, the intracell hopping between A and B is denoted by $t_1 \pm \gamma_1/2$, while the intercell hopping between site B of cell n and site A of cell $n+1$ is given by $t_2 \pm \gamma_2/2$. Additionally, an NNN hopping term with amplitude t_3 directly couples site A in cell n to site B in cell $n+1$, where NNN denotes NNN hopping across adjacent unit cells, distinct from genuine long-range interactions spanning many sites. The Hamiltonian reads

$$H = \sum_n \left[\left(t_1 - \frac{\gamma_1}{2} \right) c_{n,A}^\dagger c_{n,B} + \left(t_1 + \frac{\gamma_1}{2} \right) c_{n,B}^\dagger c_{n,A} + \left(t_2 - \frac{\gamma_2}{2} \right) c_{n,B}^\dagger c_{n+1,A} + \left(t_2 + \frac{\gamma_2}{2} \right) c_{n+1,A}^\dagger c_{n,B} + t_3 \left(c_{n,A}^\dagger c_{n+1,B} + c_{n+1,B}^\dagger c_{n,A} \right) \right], \quad (1)$$



where $c_{n,A}^\dagger$ ($c_{n,A}$) and $c_{n,B}^\dagger$ ($c_{n,B}$) are the creation (annihilation) operators for sublattices A and B in the n th unit cell, respectively. The model reduces to the conventional Hermitian SSH chain when $\gamma_1 = \gamma_2 = t_3 = 0$.

A circuit analogue of the above model is shown in figure 1(b). Each unit cell contains two nodes labeled A and B. Each node is grounded by a parallel capacitor C_0 and an inductor L_0 . Specifically, node A is grounded by a capacitor $C_0 + 2C_4$ and an inductor L_0 , while node B is grounded by a capacitor $C_0 + 2C_3$ and an inductor L_0 . The hopping amplitudes t_1 and t_2 are respectively mapped to capacitors C_1 and C_2 . The non-Hermitian parameters γ_1 and γ_2 are realized by negative-impedance converters involving capacitors C_3 and C_4 (see figure 1(d)), while the NNN hopping t_3 is represented by capacitor C_5 .

Applying Kirchhoff's and Ohm's laws to nodes A and B in cell x yields

$$I_A^x = i\omega (C_1 + C_3) (V_A^x - V_B^x) + i\omega C_5 (V_A^x - V_B^{x+1}) + i\omega (C_2 - C_4) (V_A^x - V_B^{x-1}) + \frac{1}{i\omega L_0} V_A^x + i\omega (C_0 + 2C_3) V_A^x, \quad (2)$$

$$I_B^x = i\omega (C_1 - C_3) (V_B^x - V_A^x) + i\omega C_5 (V_B^x - V_A^{x-1}) + i\omega (C_2 + C_4) (V_B^x - V_A^{x+1}) + \frac{1}{i\omega L_0} V_B^x + i\omega (C_0 + 2C_3) V_B^x, \quad (3)$$

where I_n^x and V_n^x ($n = A, B$) denote the external current and voltage at node n of cell x .

Imposing Bloch periodicity, $V_n^{x+1} = e^{ik} V_n^x$ ($n = A, B$), the circuit equations can be written in a compact Laplacian form

$$J(\omega) = i\omega [\varepsilon \mathbb{1} + H(k)], \quad (4)$$

with $\mathbb{1}$ the identity matrix, $\varepsilon = C_1 + C_2 + C_3 + C_4 + C_5 - \frac{1}{\omega^2 L_0}$, and

$$H(k) = - \begin{pmatrix} 0 & A1 + B1e^{-ik} + C_5e^{ik} \\ C1 + C_5e^{-ik} + D1e^{ik} & 0 \end{pmatrix}, \quad (5)$$

where

$$A1 = C_1 + C_3, \quad B1 = C_2 - C_4, \quad C1 = C_1 - C_3, \quad D1 = C_2 + C_4. \quad (6)$$

The eigenfrequencies of the circuit follow from the eigenvalues ε_n of $H(k)$

$$f_n = \frac{1}{2\pi \sqrt{(C_1 + C_2 + C_3 + C_4 + C_5 - \varepsilon_n) L_0}}. \quad (7)$$

Thus, the circuit Laplacian $J(\omega)$ directly maps to the Hamiltonian $H(k)$ up to an overall factor and an energy shift, establishing a faithful correspondence between the electrical circuit and the tight-binding model.

Transforming the real-space Hamiltonian in equation (1) to momentum space via the Fourier expansion

$$c_{n,\alpha} = \frac{1}{\sqrt{N}} \sum_k e^{ikn} c_{k,\alpha}, \quad \alpha = A, B, \quad (8)$$

where k is the Bloch wave vector and N is the number of unit cells, the Bloch Hamiltonian becomes

$$H(k) = \begin{pmatrix} 0 & h_{AB}(k) \\ h_{BA}(k) & 0 \end{pmatrix}, \quad (9)$$

with

$$h_{AB}(k) = \left(t_1 - \frac{\gamma_1}{2}\right) + \left(t_2 + \frac{\gamma_2}{2}\right) e^{-ik} + t_3 e^{ik}, \quad (10)$$

$$h_{BA}(k) = \left(t_1 + \frac{\gamma_1}{2}\right) + \left(t_2 - \frac{\gamma_2}{2}\right) e^{ik} + t_3 e^{-ik}. \quad (11)$$

Note that for $\gamma_1 \neq 0$ or $\gamma_2 \neq 0$, $h_{BA}(k) \neq h_{AB}^\dagger(k)$; hence $H(k)$ is non-Hermitian. The energy bands are obtained from the eigenvalues of $H(k)$

$$E(k) = \pm \sqrt{h_{AB}(k) h_{BA}(k)}. \quad (12)$$

To make the parameter mapping explicit, comparing equations (9) and (10) yields the following one-to-one correspondence between circuit capacitances and lattice parameters (up to an overall normalization factor that can be absorbed into the global energy scale)

$$t_1 = C_1, \quad \frac{\gamma_1}{2} = C_3, \quad t_2 = C_2, \quad \frac{\gamma_2}{2} = C_4, \quad t_3 = C_5. \quad (13)$$

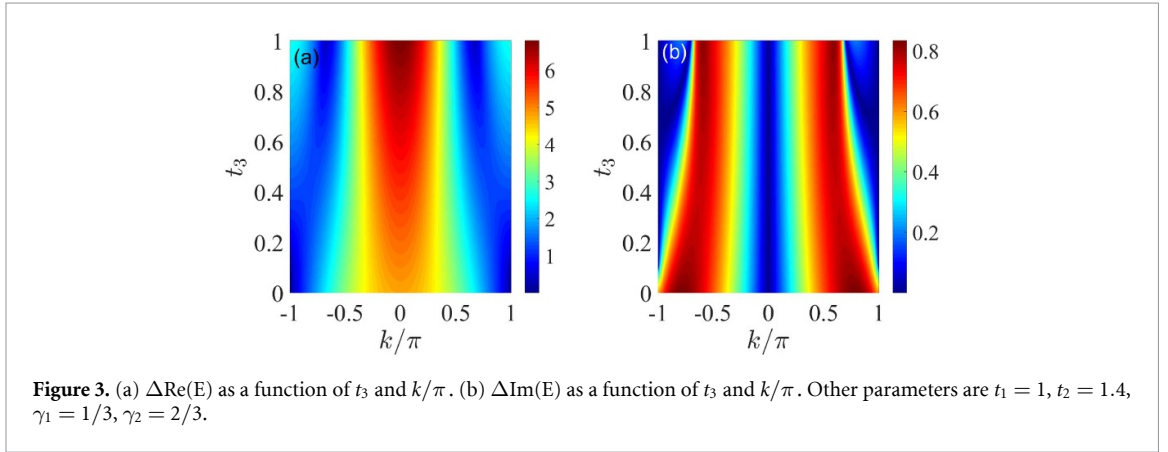
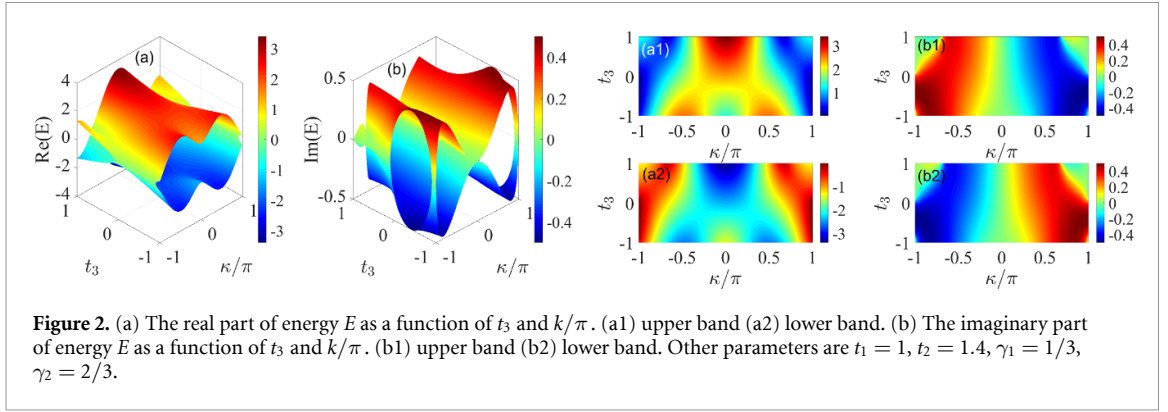
Thus, C_3 and C_4 determine the nonreciprocal hopping amplitudes $\gamma_1/2$ and $\gamma_2/2$, respectively, while C_5 directly corresponds to the NNN hopping amplitude t_3 .

We note that the non-Hermitian Hamiltonian obtained from the circuit Laplacian describes only the deterministic response of the circuit. In any real implementation, active elements such as INICs introduce noise, and the steady-state covariance is determined by both the non-Hermitian drift matrix and the noise covariance matrix. Correlation functions therefore cannot be inferred from the non-Hermitian Hamiltonian alone. Since our analysis is confined to coherent response quantities, a Langevin treatment of noise is not required. This expression will serve as the starting point for analyzing the band structure and topological properties in the following sections.

3. Band structure modulation by NNN hopping

Having established the mapping between the circuit Laplacian and the tight-binding Hamiltonian, we now examine the band structure modulation induced by NNN hopping. Based on the non-Hermitian SSH model with NNN hopping t_3 and asymmetric hoppings γ_1, γ_2 , the evolution of the real and imaginary parts of the energy bands with t_3 and wave number k in figure 2 reveals rich physics arising from the interplay between non-Hermiticity and topology. The Bloch Hamiltonian respects a chiral symmetry, and the eigenvalues are $E(k)$, where h_{AB} and h_{BA} are not complex conjugates of each other due to $\gamma_1, \gamma_2 \neq 0$, leading to a complex energy spectrum. The real parts of the bands, shown in figures 2(a1) and (a2) for the upper and lower bands respectively, directly reflect topological phase transitions. For small t_3 , a gap exists between the two bands. As t_3 increases, the gap closes at specific k points (e.g. at $k = 0$ or $k = \pi$) and then reopens, signaling a transition from a topologically nontrivial phase to a trivial one. The NNN hopping amplitude t_3 modifies the dispersion through the $e^{\pm ik}$ factors in h_{AB} and h_{BA} , thereby controlling the location of gap closing and the curvature of the bands.

The imaginary parts of the bands, presented in figures 2(b1) and (b2), manifest the non-Hermitian nature. Because γ_1 and γ_2 break reciprocity, the product $h_{AB}(k)h_{BA}(k)$ becomes complex, giving rise to nonzero imaginary parts of the energy, whose signs indicate loss (negative) or gain (positive). The alternating signs of the imaginary part across k space indicate that the system is in a PT symmetry broken phase, where eigenstates appear in complex-conjugate pairs. When the imaginary part vanishes at



some k as t_3 varies, an exceptional point is encountered, where the real bands also become degenerate. Exceptional points are singularities unique to non-Hermitian systems and play a crucial role in topological transitions and sensing applications. Thus, the parameter t_3 plays a dual role, it tunes the topological structure of the real bands by modifying the phase factors in h_{AB} and h_{BA} , and it also controls the distribution of the imaginary parts through its coupling with γ_1, γ_2 , thereby affecting the degree of PT symmetry breaking. The coordinated evolution of the real and imaginary bands with t_3 and k in figure 2 provides a clear picture of the interrelation among band topology, exceptional points, and PT-symmetry breaking in non-Hermitian topological systems.

Figure 3 presents the differences between the upper and lower energy bands, defined as $\Delta\text{Re}(E)$ and $\Delta\text{Im}(E)$, as functions of the NNN hopping amplitude t_3 and the wave number k/π . These quantities directly characterize the gap structure and the degeneracies in the complex energy spectrum, providing complementary information to the individual band structures shown in figure 2. The real part difference $\Delta\text{Re}(E)$, displayed in figure 3(a), reveals the locations where the real parts of the two bands become degenerate. For small t_3 , $\Delta\text{Re}(E)$ is nonzero over most of the Brillouin zone, indicating a finite real energy gap. As t_3 increases, regions where $\Delta\text{Re}(E) = 0$ emerge, particularly near $k/\pi = 0$ and $k/\pi = 1$. These gap-closing lines are hallmarks of topological phase transitions: when the system parameters cross such a line, the real bands close and then reopen, typically accompanied by a change in the topological invariant. The NNN hopping t_3 controls the positions and the number of these gap-closing points by modifying the phase factors in $h_{AB}(k)$ and $h_{BA}(k)$.

The imaginary part difference $\Delta\text{Im}(E)$, shown in figure 3(b), reflects the non-Hermitian nature of the system. Since γ_1 and γ_2 break reciprocity, the energies acquire nonzero imaginary parts, and $\Delta\text{Im}(E)$ captures how the imaginary parts of the two bands differ across parameter space. The alternating positive and negative values of $\Delta\text{Im}(E)$ indicate that the system resides in a PT-symmetry broken phase, where eigenstates appear in complex conjugate pairs. Notably, the intersections between the zero lines of $\Delta\text{Re}(E)$ and $\Delta\text{Im}(E)$ correspond to exceptional points, where both the real and imaginary parts of the two bands become degenerate simultaneously. These exceptional points are non-Hermitian singularities that govern the system sensitivity and topological properties.

The above analysis of the band differences reveals the locations of gap closing and exceptional points. The phase transition process is further clarified by calculating the density of states (DOS) distribution of the real and imaginary parts of the energy bands as a function of the NNN hopping parameter t_3 . It

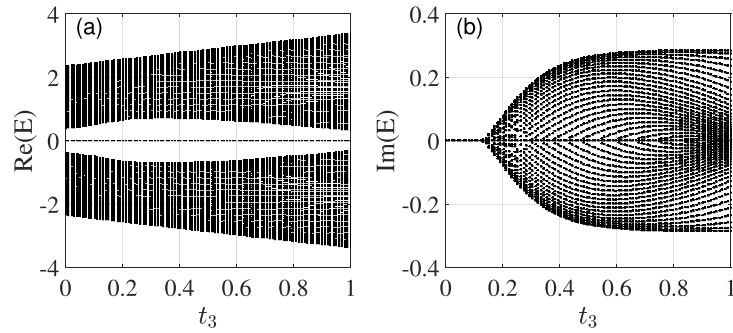


Figure 4. (a) DOS distribution of the real part of the band as a function of t_3 . (b) DOS distribution of the imaginary part of the band as a function of t_3 . Other parameters are $t_1 = 1$, $t_2 = 1.4$, $\gamma_1 = 1/3$, $\gamma_2 = 2/3$.

should be noted that the DOS here is not the conventional DOS, but rather a statistical measure used to characterize the density of eigenvalues within a specific energy range. Figure 4(a) shows the DOS distribution of the real part of the energy bands. As can be seen, the distribution of the real part does not change significantly with increasing t_3 , the real energy gap remains present throughout, with only minor variations in intensity. This indicates that the NNN hopping t_3 has a limited effect on the structure of the real part of the bands. Within the parameter range examined, no real-gap-closing-driven topological phase transition is observed.

In stark contrast, figure 4(b) displays the DOS distribution of the imaginary part of the energy bands. For small t_3 , the imaginary-part DOS is concentrated near zero, indicating that the system resides in the PT-symmetric phase with a purely real energy spectrum. As t_3 increases, the DOS gradually shifts toward nonzero imaginary energy regions, signaling the system transition into the PT-symmetry broken phase, where the energy spectrum becomes complex. This clear evolution from the zero-valued region to the nonzero region vividly demonstrates the system transition from the PT-symmetric phase to the PT-symmetry broken phase. Moreover, the behavior near the critical point of the phase transition can be clearly identified, providing a visual basis for further analysis of the phase transition mechanism.

4. Spectral and localization properties under OBCs

4.1. Admittance and energy spectra under OBC and PBC

We investigate a finite non-Hermitian SSH circuit with $N = 40$ nodes, which incorporates NNN hopping terms. The circuit Laplacian is given by

$$J(\omega) = \begin{pmatrix} M & -J_1 & 0 & -J_5 & 0 & \cdots \\ -J_2 & M & -J_3 & 0 & 0 & \cdots \\ 0 & -J_4 & M & -J_1 & 0 & \cdots \\ -J_5 & 0 & -J_2 & M & -J_3 & 0 \cdots \\ 0 & 0 & 0 & -J_4 & M & -J_1 \cdots \\ 0 & 0 & -J_5 & 0 & -J_2 & M \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{N \times N}, \quad (14)$$

where $M = i\omega(C_0 + C_1 + C_2 + C_3 + C_4 + C_5) - i/(\omega L_0)$, $J_1 = i\omega(C_1 + C_3)$, $J_2 = i\omega(C_1 - C_3)$, $J_3 = i\omega(C_2 + C_4)$, $J_4 = i\omega(C_2 - C_4)$, and $J_5 = i\omega C_5$. To clearly observe the effect of NNN hopping in a non-Hermitian setting, we choose the parameters $C_0 = 470$ pF, $C_1 = 300$ pF, $C_2 = 680$ pF, $C_3 = 100$ pF, $C_4 = 200$ pF, and $L_0 = 47$ μ H. Under OBCs, we compute the admittance spectrum and eigenfrequency spectrum of the non-Hermitian SSH chain and systematically examine how varying the NNN hopping capacitance C_5 affects these spectra, as shown in figure 5. Our results demonstrate that introducing NNN hopping significantly modulates the topological and localization properties of the system.

For $C_5 = 0$ (figures 5(a1) and (b1)), the circuit exhibits the characteristic features of a standard non-Hermitian SSH model. A pair of isolated in-gap modes appears in the admittance spectrum, corresponding to topological edge states, while the bulk eigenstates are strongly localized at the right boundary a clear signature of the NHSE. This behavior originates from the non-reciprocal hopping terms γ_1, γ_2 (implemented via C_3 and C_4), which induce exponential boundary accumulation of bulk states under OBC, while chiral symmetry protects the topological edge states. As C_5 increases to 300 pF (figures 5(a2)

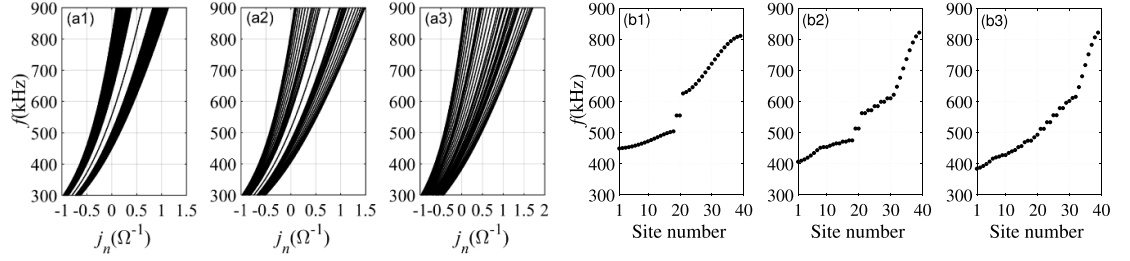


Figure 5. Evolution of the admittance spectrum and the spatial distribution of eigenstates of the non-Hermitian SSH circuit with increasing NNN hopping capacitance C_5 . The admittance spectra for (a1) $C_5 = 0$, (a2) $C_5 = 300$ pF, and (a3) $C_5 = 500$ pF. The corresponding eigenstate distributions for the same parameters are shown in (b1)–(b3). Other circuit parameters are $C_0 = 470$ pF, $C_1 = 300$ pF, $C_2 = 680$ pF, $C_3 = 100$ pF, $C_4 = 200$ pF, and $L_0 = 47$ μ H.

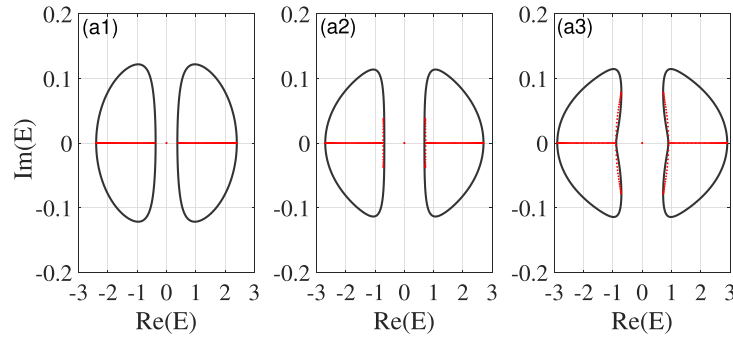
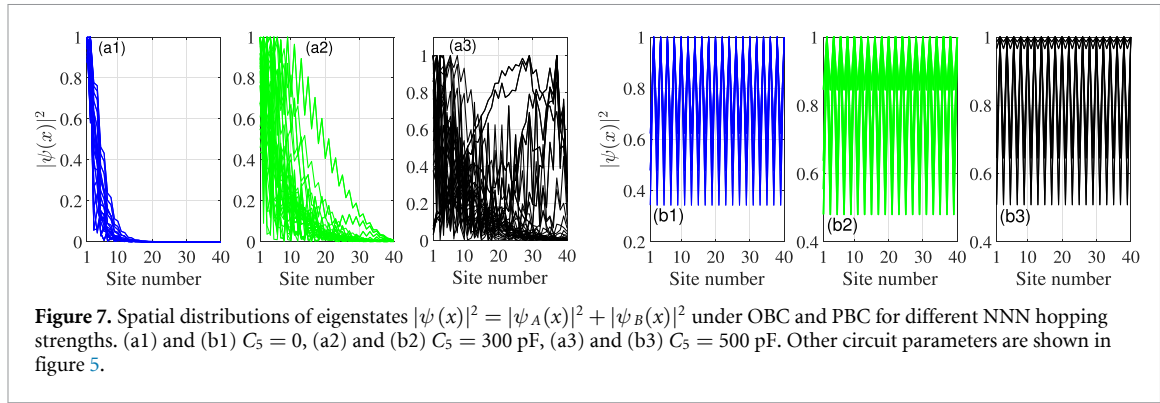


Figure 6. Energy spectra of the non-Hermitian SSH model under PBC (black) and OBC (red) for increasing NNN hopping capacitance C_5 . The spectra are shown for (a1) $C_5 = 0$, (a2) $C_5 = 300$ pF, and (a3) $C_5 = 500$ pF. Other circuit parameters are shown in figure 5.

and (b2)), the NNN hopping t_3 (associated with C_5) preserves chiral symmetry (since it connects opposite sublattices) while providing additional channels that enhance bulk connectivity. Consequently, the edge states, though topologically protected, hybridize with bulk modes, and the isolated peaks in the admittance spectrum broaden and weaken, and the spatial profiles of the eigenstates show noticeable delocalization toward the interior of the chain. This indicates that NNN hopping suppresses the NHSE by providing additional hopping pathways that reduce boundary-induced localization. At $C_5 = 500$ pF (figures 5(a3) and (b3)), the in-gap features in the admittance spectrum nearly vanish and the band gap closes. Simultaneously, the eigenstates become almost uniformly extended throughout the lattice, demonstrating that the NHSE is strongly suppressed. This evolution reflects a continuous transition from a boundary-localized non-Hermitian phase to a bulk-extended, Hermitian-like regime, consistent with the theoretical prediction that the GBZ contracts toward the unit circle as the NNN hopping strength increases.

Further insight into the role of boundary conditions is gained by comparing the energy spectra under PBCs and OBC for different values of C_5 , as displayed in figure 6. In the absence of NNN hopping ($C_5 = 0$, figure 6(a1)), the PBC (black) and OBC (red) spectra deviate markedly in the complex plane. The OBC spectrum contracts and distorts relative to the PBC one a hallmark of the NHSE, where bulk states under open boundaries accumulate exponentially at the edges, shifting their eigenenergies away from the Bloch bands. With C_5 increased to 300 pF (figure 6(a2)), the PBC and OBC spectra move closer together and their shapes become more similar, indicating reduced boundary sensitivity and a suppression of the NHSE. When C_5 reaches 500 pF (figure 6(a3)), the spectra under the two boundary conditions nearly coincide and the band gap almost closes, signaling a transition into a bulk-dominated regime that is largely insensitive to boundary conditions. This behavior corresponds to the contraction of the GBZ toward the unit circle and aligns with the critical region of a topological phase transition defined by non-Bloch invariants. By tuning the NNN hopping capacitance C_5 , we can continuously control the topological and localization properties of the non-Hermitian SSH system. The evolution of both the admittance and energy spectra presents a consistent physical picture: enhanced NNN hopping improves bulk connectivity without breaking chiral symmetry, and drives the system from a boundary-sensitive non-Hermitian topological phase toward a bulk-extended, Hermitian-like regime. This process



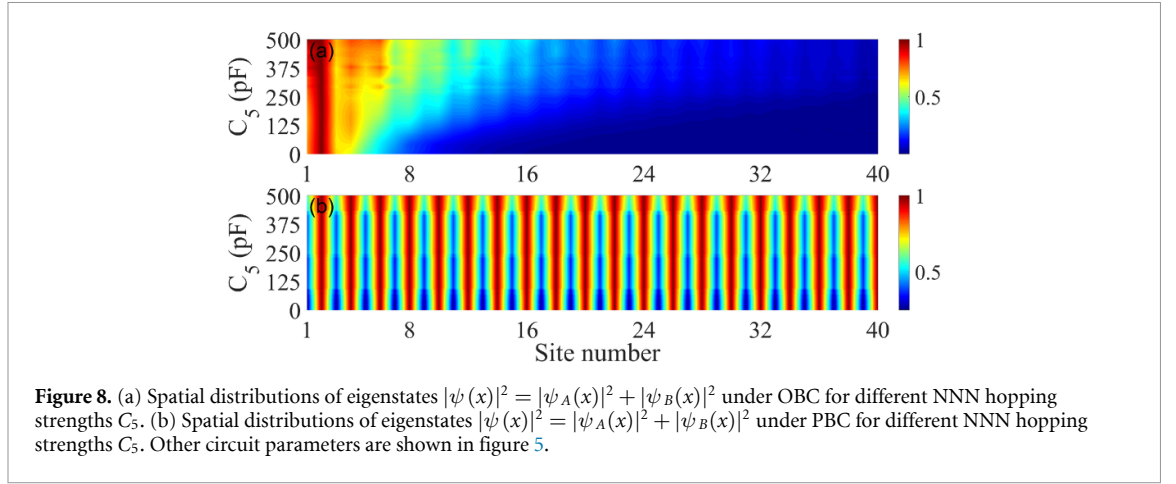
reflects the contraction of the GBZ toward the unit circle. Our findings highlight the potential of a single circuit parameter, C_5 , to dynamically tune between distinct topological and localization phases, offering a versatile platform for exploring non-Hermitian topology in an electrical-circuit context. These spectral features provide strong evidence that NNN hopping reduces the sensitivity of the system to boundary conditions. To directly visualize this suppression of the NHSE, we next examine the spatial distribution of eigenstates.

4.2. Spatial delocalization of eigenstates

To directly visualize and quantify the suppression of the NHSE, we examine the spatial distribution of eigenstates and the circuit impedance response under varying NNN hopping. The most representative phenomenon in non-Hermitian systems is the NHSE, characterized by the exponential decay of system eigenstates localized at boundaries. This phenomenon is ubiquitous in non-Hermitian systems, demonstrating their sensitivity to boundary conditions. Such sensitivity causes the system wave function to exhibit fundamentally different behaviors under periodic and open boundary conditions.

We next examine how NNN hopping modifies this effect. Figure 7 displays the eigenstate probability density, $|\psi(x)|^2 = |\psi_A(x)|^2 + |\psi_B(x)|^2$, for the non-Hermitian SSH model under OBC and PBC for different NNN capacitances C_5 , directly revealing how NNN hopping regulates the NHSE. For $C_5 = 0$ (figures 7(a1) and (b1)), the signature NHSE is evident, under OBC, eigenstates are strongly localized at one end of the chain, whereas under PBC, they are extended uniformly across the lattice. This stark contrast exemplifies the boundary-induced exponential localization characteristic of the NHSE. As C_5 increases to 300 pF (figures 7(a2) and (b2)), the NNN hopping becomes significant. Under OBC, the boundary-localized peaks diminish and states spread into the bulk, indicating a weakened skin effect. Under PBC, the extended profile persists but acquires mild modulation. These changes show that NNN hopping enhances inter-site connectivity, thereby suppressing the boundary attraction responsible for the NHSE. For $C_5 = 500$ pF (figures 7(a3) and (b3)), the probability distributions under OBC and PBC become nearly identical, both exhibiting almost uniform extension throughout the system. Here, the NHSE is strongly suppressed, and the system sensitivity to boundary conditions is drastically reduced. This marks a transition driven by NNN hopping from a boundary localized non-Hermitian phase to a bulk-extended, Hermitian-like phase. From the perspective of real-space wavefunctions, figure 7 demonstrates that NNN hopping can progressively weaken and nearly eliminate the NHSE by enhancing bulk connectivity, without altering the chiral symmetry of the model. This shifts the system behavior from being dominated by boundary sensitivity to being governed by bulk properties. These observations are consistent with and reinforce the spectral and state-distribution results in figures 5 and 6, collectively providing a coherent picture of how NNN hopping tailors non-Hermitian topological states.

Under open boundary conditions, as shown in figure 8(a), the eigenstate distributions display strong inhomogeneity. When the NNN hopping C_5 is small, the eigenstates are predominantly localized at one end of the system (e.g. the left boundary), which is a hallmark of the NHSE a phenomenon where non-Hermiticity causes a large number of eigenstates to become exponentially localized at the boundary. As C_5 increases, the localization direction shifts toward the opposite end (the right boundary), demonstrating that C_5 can effectively tune both the direction and intensity of the skin effect. In contrast, under periodic boundary conditions, as shown in figure 8(b), the eigenstate distributions are nearly uniform, exhibiting typical Bloch-wave characteristics. This indicates that the system maintains translational invariance under PBC, with all eigenstates extended across the entire lattice. The striking contrast between the strongly asymmetric eigenstate distributions under OBC and the uniform distributions



under PBC directly reveals the extreme sensitivity of non-Hermitian energy spectra to boundary conditions, as well as the complex interplay between bulk and boundary states.

Further analysis indicates that C_5 , as the NNN hopping strength, not only modifies the distribution of the imaginary parts of the energy bands (as shown in figures 2 and 3) but also controls the direction of the NHSE by modulating the effective non-reciprocal hopping strength. Under OBC, the transition of eigenstate localization from one end to the other signals a possible topological transition in the skin effect, characterized by a change in the winding number or skin direction of the non-Hermitian energy bands. This behavior is consistent with the evolution of the imaginary-part DOS in figure 4, which shows a transition from the zero-valued region to nonzero regions, collectively illustrating that C_5 drives both the PT-symmetry phase transition and the real-space localization behavior of eigenstates. The delocalization of eigenstates observed above indicates a fundamental change in the system's response to boundaries. A complementary probe of this transition is the impedance response, which directly reflects the spatial distribution of circuit modes.

4.3. Impedance response as a probe of localization

The influence of NNN hopping terms on the NHSE is further validated by computing the system impedance. The characteristic value and characteristic vector of Laplace matrix J can be used to represent the impedance between any two nodes in the resistance network. The Laplacian J is a real symmetric $N \times N$ matrix with N real eigenvalues λ_n and an orthonormal basis of eigenvectors $\psi_n = (\psi_{n1}, \dots, \psi_{nN})^\dagger$. The impedance between the nodes A and B is given by [68]

$$Z_{AB} = \sum_{n=1}^N \frac{1}{\lambda_n (\psi_n^l)^* \psi_n^r} (\psi_{nA}^l - \psi_{nB}^l) (\psi_{nA}^r - \psi_{nB}^r), \quad (15)$$

where l and r correspond to the left and right eigenvectors, respectively. In the circuit system, according to Ohm's law, the impedance between nodes A and B is $Z_{AB} = (V_A - V_B)/I$. Then, according to the Green function, we get the expression of impedance in the circuit system as (see equations (A11) in appendix A for details)

$$Z_{AB} = \sum_{n=1}^N \frac{1}{j_n (V_n^l)^* V_n^r} (V_{nA}^l - V_{nB}^l) (V_{nA}^r - V_{nB}^r), \quad (16)$$

The formula connects the impedance between any two nodes in the circuit network with the eigenvalue and eigenvector of the Laplace matrix, so the impedance can reflect the intrinsic characteristics of the circuit system. When $j_n = 0$, $Z_{AB} \rightarrow \infty$, which means the impedance has a resonance peak at the natural frequency of the system. Notably, the resonance peak corresponds one-to-one with the eigenstates of the circuit. To compare impedance distributions across different coupling strengths, it is necessary to eliminate absolute scale differences. Accordingly, we normalize the impedance magnitude in both theoretical calculations and circuit simulations, defining the normalized impedance as

$$|Z|_{\text{norm}} = \frac{|Z|}{|Z|_{\text{max}}}, \quad (17)$$

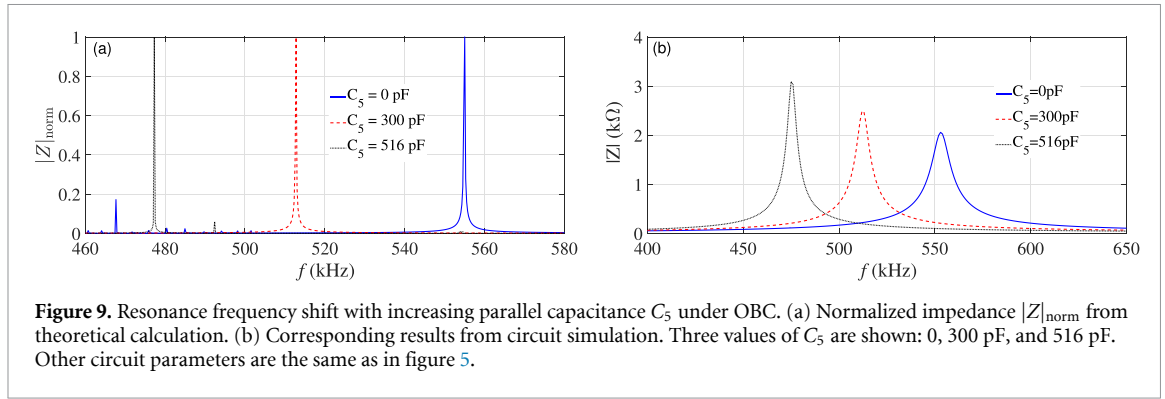


Figure 9. Resonance frequency shift with increasing parallel capacitance C_5 under OBC. (a) Normalized impedance $|Z|_{\text{norm}}$ from theoretical calculation. (b) Corresponding results from circuit simulation. Three values of C_5 are shown: 0, 300 pF, and 516 pF. Other circuit parameters are the same as in figure 5.

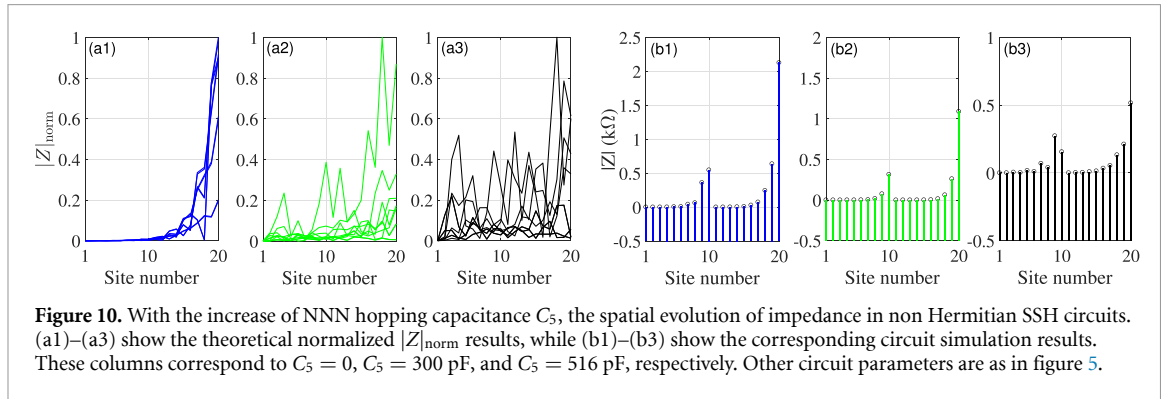


Figure 10. With the increase of NNN hopping capacitance C_5 , the spatial evolution of impedance in non Hermitian SSH circuits. (a1)–(a3) show the theoretical normalized $|Z|_{\text{norm}}$ results, while (b1)–(b3) show the corresponding circuit simulation results. These columns correspond to $C_5 = 0$, $C_5 = 300$ pF, and $C_5 = 516$ pF, respectively. Other circuit parameters are as in figure 5.

where $|Z|_{\text{max}}$ is the maximum impedance magnitude among all nodes in the system for a given set of parameters. This normalization procedure eliminates the absolute scale differences and highlights the spatial variation of the impedance, facilitating a direct comparison of the localization and delocalization behavior under different coupling strengths.

With this normalization, figure 9 illustrates the variation of the system resonance frequency with increasing parallel capacitance C_5 under OBC. Figures 9(a) and (b) clearly indicate that as C_5 increases from 0 to 300 pF and further to 516 pF, the resonance peak shifts systematically toward lower frequencies. This behavior is consistent with the fundamental resonance formula $f_r = 1/(2\pi\sqrt{LC_{\text{eff}}})$, where the effective capacitance C_{eff} includes the contribution of C_5 . The increase in C_5 raises C_{eff} , thereby reducing the resonance frequency. The three curves, ordered from right to left in terms of increasing frequency, correspond to decreasing values of C_5 , confirming the continuous tunability of the resonance frequency through the parallel capacitance.

From a physical perspective, the parallel capacitance C_5 , when placed across the resonant structure under OBC, directly loads the terminal and modifies the total capacitive energy storage of the system. Consequently, variations in C_5 not only shift the resonance frequency but also alter the peak impedance amplitude, as evident in the figure. This reflects changes in the energy storage and loss characteristics at resonance, thereby modulating the input impedance of the system. The close agreement between theoretical calculations and circuit simulations validates the accuracy of the underlying physical model. The theoretical curves are derived from an ideal circuit model that analytically solves the impedance frequency response, while the simulations incorporate practical factors such as parasitic elements and layout effects. The consistent trends and curve shapes confirm that the tuning mechanism via C_5 is reliable and practically realizable. Minor discrepancies between theory and simulation attributable to non-ideal components (e.g. parasitic capacitances, equivalent series resistance), stray couplings, numerical tolerances in simulation software, or potential experimental noise do not affect the overall physical trend. These factors are inherent in any practical realization and do not alter the main conclusion regarding the suppression of the NHSE by NNN hopping.

In addition, we investigate the effect of NNN hopping on impedance and plotted the impedance distribution of each node under different coupling intensities. Figure 10 illustrates the systematic evolution of the spatial impedance distribution in the non-Hermitian SSH circuit under OBC as the NNN hopping capacitance C_5 increases. The change in the impedance profile provides direct and quantitative evidence for the suppression of the NHSE by NNN hopping, revealing the system transition from a boundary-localized to a bulk-extended phase. For $C_5 = 0$ (figures 10(a1) and (b1)), the impedance is

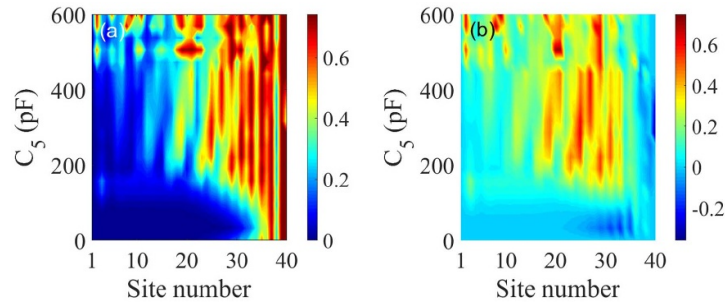


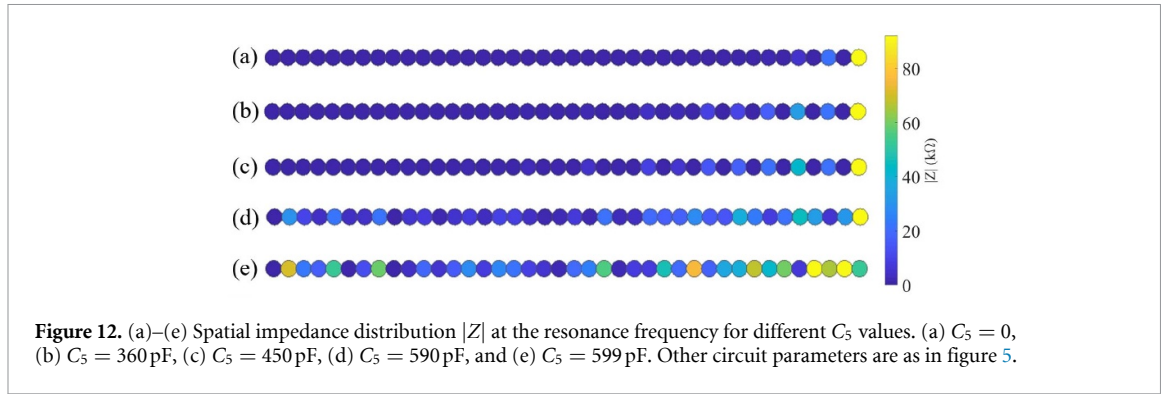
Figure 11. (a) Impedance $|Z|$ distribution as a function of C_5 and Site number. (b) The impedance difference $\Delta|Z|$ distribution as a function of C_5 and Site number. Other circuit parameters are shown in figure 5.

strongly localized near the right boundary of the chain, exhibiting a sharp spatial peak. This concentration coincides with the boundary accumulation of eigenstates under the NHSE, a consequence of the non-Hermitian hopping terms that cause exponential localization of bulk wavefunctions at the edge. The impedance resonance thus reflects the same skin modes that appear in the admittance and eigenstate spectra. The close agreement between theory and simulation confirms the faithful mapping between the circuit Laplacian and the tight-binding Hamiltonian. When C_5 is raised to 300 pF (figures 10(a2) and (b2)), the impedance peak broadens and its height decreases substantially; the region of elevated impedance extends from the boundary into the interior of the lattice. This delocalization indicates that the NNN hopping provides additional hopping paths that reduce the boundary dominance, while preserving the chiral symmetry of the model. The flattening of the impedance profile mirrors the spatial diffusion of eigenstates observed, demonstrating the progressive weakening of the NHSE. At the largest coupling strength, $C_5 = 516$ pF (figures 10(a3) and (b3)), the impedance becomes nearly uniform across the entire chain. The boundary-localized resonance has almost completely vanished, and the system behaves like a homogeneous bulk network. This near-uniform distribution signals that the NHSE is strongly suppressed and that the influence of the open boundaries has become negligible. The circuit thus enters a Hermitian-like extended regime, driven entirely by the NNN hopping.

To more accurately reveal the impact of NNN hopping on the impedance characteristics of the system, we first define the impedance difference $\Delta|Z|$ to quantify the change in the impedance magnitude at a node before and after connecting the capacitor C_5

$$\Delta|Z| = |Z_i(C_5)| - |Z_i(0)|, \quad (18)$$

where $|Z_i(C_5)|$ is the impedance magnitude at node i after connecting the capacitor C_5 , and $|Z_i(0)|$ is the impedance magnitude at the same node without the capacitor (i.e. $C_5 = 0$). This difference provides a direct measure of the net effect of NNN hopping on the impedance at each node. Figure 11 presents the distribution of the node impedance $|Z|$ (figure 11(a)) and the impedance difference $\Delta|Z|$ (figure 11(b)) as functions of the NNN hopping strength C_5 , offering real-space evidence for understanding the competition between the NHSE and NNN hopping. From figure 11(a), it can be observed that when C_5 is small, the impedance magnitude is predominantly concentrated at the right side of the system, exhibiting clear boundary localization. This is a typical manifestation of the NHSE, under open boundary conditions, a large number of eigenstates become exponentially localized at the boundary, leading to a significantly enhanced impedance response at that boundary. As C_5 gradually increases, the impedance distribution, originally localized on the right side, begins to extend into the bulk region. The impedance peak at the right boundary weakens, while the impedance values at bulk nodes increase. This evolution clearly demonstrates a competitive relationship between NNN hopping and the NHSE, NNN hopping, by introducing additional long-distance hopping channels, weakens the boundary localization induced by non-Hermiticity, shifting the system response from boundary localization to a bulk-distributed profile. Figure 11(b) further reveals the specific influence of NNN hopping on the system impedance through the impedance difference $\Delta|Z|$. When C_5 is small, $\Delta|Z|$ is close to zero for most nodes, indicating that the perturbation from NNN hopping is minor and the system remains dominated by the NHSE. As C_5 increases, regions where $\Delta|Z|$ was originally zero gradually develop nonzero distributions, and the patterns of positive and negative values are highly consistent with the trend of impedance variation shown in figure 11(a). This evolution quantitatively demonstrates that the introduction of NNN hopping significantly alters the impedance characteristics of the system, and its effect progressively extends from



boundary nodes to bulk nodes as C_5 increases, further confirming the competitive relationship between NNN hopping and the NHSE.

To more intuitively reflect the influence of NNN hopping term on impedance, i.e. the weakening effect of non-Hermite skin effect, we draw the spatial distribution of impedance at the resonance frequency. Figure 12 demonstrates the crucial role of the NNN hopping term C_5 in modulating the spatial impedance distribution and suppressing the NHSE in a non-reciprocal circuit chain. As C_5 increases from 0 to $C_5 = 599$ pF, the system undergoes a complete phase transition from a strong NHSE state to a uniform bulk state. When $C_5 = 0$ (figure 12(a)), the impedance is highly localized at the right boundary of the system, exhibiting the characteristic signature of the NHSE. This phenomenon originates from the non-Bloch band topology of non-Hermitian systems, where eigenstates with complex wavevectors are defined on a generalized Brillouin zone, leading to exponentially localized wavefunctions in real space. The impedance concentration follows an exponential decay profile $Z(x) \sim Z_0 e^{-x/\xi}$, where Z_0 represents the impedance amplitude at the boundary and ξ is the localization length determined by the non-reciprocal couplings C_3 and C_4 .

As C_5 gradually increases to $C_5 = 360$ pF and $C_5 = 450$ pF (figures 12(b) and (c)), the impedance distribution undergoes significant changes: the initially boundary-localized impedance progressively delocalizes and diffuses into the bulk, while the impedance magnitude at interior nodes rises accordingly. This transformation corresponds to the partial suppression of the NHSE, whose physical mechanism lies in the fact that the NNN hopping C_5 introduces additional multi-path interference effects that disrupt the directionality of the original non-reciprocal coupling. When C_5 reaches $C_5 = 590$ pF and $C_5 = 599$ pF (figures 12(d) and (e)), the impedance distribution exhibits nearly uniform bulk characteristics, indicating the complete suppression of the NHSE. Near the critical point $C_5^c \approx C_5 = 590$ pF, the system exhibits rich scaling behavior and finite-size effects. The impedance non-uniformity parameter $\chi = (\max |Z| - \min |Z|) / \langle |Z| \rangle$ diverges with a power-law dependence, $\chi \sim |C_5 - C_5^c|^{-\gamma}$, $\gamma \approx 0.67$, characteristic of a continuous phase transition. The localization length ξ simultaneously diverges as $\xi \sim |C_5 - C_5^c|^{-\nu}$ with $\nu \approx 1.0$, consistent with theoretical predictions for non-Hermitian topological phase transitions. This controllable phase transition process not only validates fundamental predictions of non-Hermitian topological theory but also provides a physical foundation for designing tunable non-reciprocal devices. The ability to dynamically suppress the NHSE through a single parameter adjustment demonstrates the great potential of classical circuit platforms for simulating complex quantum phenomena and realizing reconfigurable topological metamaterials. While the impedance profiles provide a qualitative visualization of the delocalization process, a quantitative characterization is necessary to capture the transition more precisely. We therefore introduce the inverse participation rate (IPR) as a measure of eigenstate localization.

4.4. IPR and multi-parameter competition

In non-Hermitian systems, the NHSE drives topological boundary states to localize at the system edge. As the strength of the NNN hopping increases, these boundary states gradually diffuse into the bulk region. This phenomenon indicates that NNN hopping can promote a transition from localized to extended states. To test this hypothesis, we introduce the IPR as a quantitative measure of spatial localization, defined as

$$\text{IPR} = \sum_{n=1}^L \sum_{\sigma \in \{A, B\}} |\psi(n, \sigma)|^4, \quad (19)$$

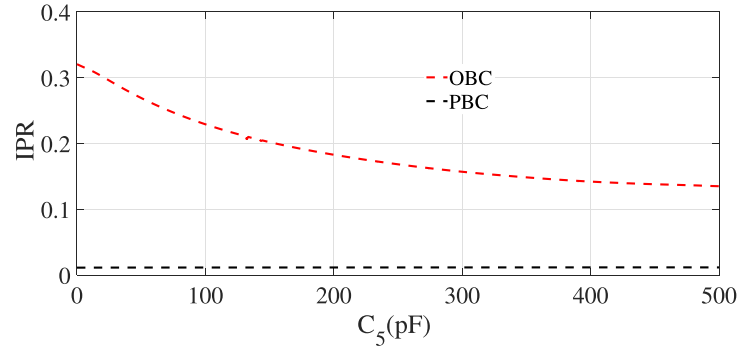


Figure 13. Evolution of the IPR with the NNN hopping capacitance C_5 under OBC (red) and PBC (black). Other circuit parameters are as in figure 5.

where L is the total number of unit cells, $\sigma = A, B$ denotes the sublattice index, and $\psi(n, \sigma)$ is the normalized eigenfunction amplitude on sublattice σ of the n th cell. If the wave function is uniformly extended over the whole system, the IPR becomes small. Specifically, for a state perfectly localized on a single site, $\text{IPR} \rightarrow 1$; for a perfectly uniform extended state, $\text{IPR} \sim 1/L \rightarrow 0$ in the thermodynamic limit. Hence, a larger IPR indicates stronger real-space localization, whereas a smaller IPR reflects a more delocalized (extended) state. By computing and analyzing the IPR under various NNN hopping strengths, we can quantitatively trace the evolution of the localization-delocalization transition in non-Hermitian systems, thereby revealing how NNN hopping modulates the NHSE. The distinct IPR behaviors under OBC and PBC originate from the fundamentally different nature of eigenstates in non-Hermitian systems. Under PBC, translational symmetry enforces Bloch waves $\psi_n \sim e^{ikn}$ with k real, resulting in uniformly extended states and $\text{IPR} \sim 1/L \rightarrow 0$. Under OBC, the broken translational symmetry allows exponential modes $\psi_n \sim \beta^n$ with $|\beta| \neq 1$, leading to boundary localization, i.e. the NHSE. The deviation of $|\beta|$ from unity is quantified by the deformation of the GBZ away from the unit circle. As C_5 increases, the GBZ contracts toward the unit circle, reducing this deviation and thus monotonically decreasing the IPR. The variation of the IPR with the NNN hopping capacitance C_5 is displayed in figure 13 for both OBC and PBC. The IPR quantifies the spatial localization of eigenstates, where a larger value indicates stronger real-space localization. Under PBC, the IPR remains close to zero for all C_5 , reflecting the extended Bloch-like character of eigenstates in the absence of the NHSE. In contrast, under OBC the IPR starts at a high value (approximately 0.32) when $C_5 = 0$, signaling the pronounced boundary localization characteristic of the NHSE. As C_5 increases, the OBC-IPR decreases monotonically, dropping to about 0.12 at $C_5 = 500$ pF. This continuous decline quantitatively demonstrates that enhanced NNN hopping progressively suppresses the NHSE, driving the system from a boundary-localized regime toward a bulk-extended one.

The suppression of the NHSE can be understood through the lens of non-Bloch band theory. In our model, the bulk wavefunctions under OBC are superpositions of terms $\psi_n \sim \beta^n$, where the complex modulus $|\beta|$ deviates from unity in the presence of non-reciprocal couplings. The NHSE strength is directly tied to this deviation, with $|\beta| \neq 1$ causing exponential boundary localization. Introducing NNN hopping t_3 (implemented by C_5) modifies the characteristic equation $\det[\mathcal{H}(\beta) - E] = 0$ of the generalized Bloch Hamiltonian. As t_3 increases, the relevant solutions β approach the unit circle ($|\beta| \rightarrow 1$), reducing the wavefunction exponential decay rate and thus suppressing boundary localization. Concurrently, from a band structure perspective, the NNN hopping provides additional hopping pathways that allow particles to bypass the boundary, causing the OBC energy spectrum to approach the PBC spectrum and reducing sensitivity to boundaries. While this coupling provides additional hopping pathways that compete with non-reciprocal hopping to suppress the NHSE, it does not break the chiral symmetry protecting the topological edge states. The primary mechanism for delocalization is the enhanced bulk connectivity, which facilitates wavefunction diffusion into the interior. Therefore, NNN hopping suppresses the NHSE by reshaping the GBZ (bringing it closer to the unit circle) and reducing the exponential localization of wavefunctions. This physical picture is consistently verified by multiple observables: the decrease in the IPR (figure 13), the evolution of eigenstate distributions from boundary-localized to bulk-extended (figure 7), and the homogenization of the spatial impedance profile (figures 10 and 12). These results collectively outline how NNN hopping modulates non-Hermitian topological states and localization behavior. The above investigations have focused on the influence of NNN hopping on specific spectral and spatial observables. To gain a comprehensive understanding of its

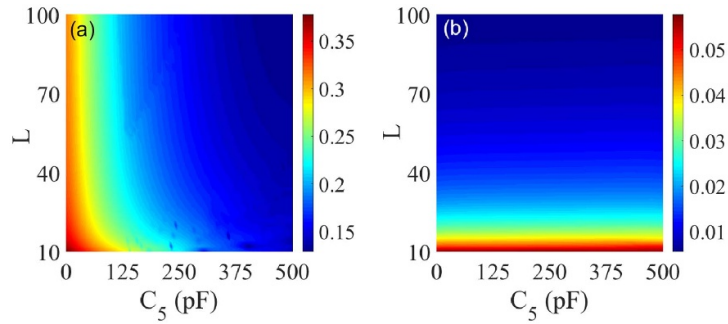


Figure 14. (a) IPR as a function of C_5 and L under OBC. (b) IPR as a function of C_5 and L under PBC. Other circuit parameters are as in figure 5.

impact on the global topological properties, a systematic analysis using topological invariants is required. In the following section, we therefore investigate the physical mechanism through which NNN hopping influences these invariants.

While previous studies have primarily focused on fixed system sizes, revealing that the IPR decreases with increasing NNN hopping strength, we extend the analysis to a range of system sizes. To investigate whether the same phenomenon exists under larger or smaller system sizes (L), we analyzed the variation of IPR with increasing NNN hopping strength under open boundary and periodic boundary conditions for different system sizes, as shown in figure 14. Under open boundary conditions, as shown in figure 14(a), when the NNN hopping C_5 is small, the IPR remains high across the entire range of system sizes and exhibits little dependence on L . This behavior indicates that the system is in a phase dominated by the NHSE, a large number of eigenstates become exponentially localized at the boundary, with a localization length much smaller than the system size, so that the IPR approaches a finite value even as L increases. As C_5 gradually increases, the IPR decreases significantly and begins to show a decreasing trend with increasing L . This suggests that NNN hopping, by introducing additional hopping channels, effectively weakens the boundary localization induced by the NHSE, causing eigenstates to extend into the bulk and reducing the degree of localization. When C_5 becomes sufficiently large, the IPR tends to zero as L increases, indicating that the system has entered an extended phase where the NHSE is completely suppressed by NNN hopping.

In contrast, under PBCs, as shown in figure 14(b), the IPR decreases significantly with increasing L across the entire range of C_5 values, and the numerical values are much smaller than those under OBC. This behavior is consistent with the expectation that eigenstates under PBC are Bloch waves: in a translationally invariant system, eigenstates are extended throughout the lattice, and the IPR scales to zero as the system size increases. Notably, even as C_5 varies, the IPR under PBC consistently maintains this extended-state characteristic, confirming that non-Hermiticity does not induce eigenstate localization under PBCs, the skin effect is a phenomenon unique to open boundary conditions. Comparing the behaviors in figures 14(a) and (b), the following physical mechanism can be identified: the eigenstate localization in the non-Hermitian SSH circuit model is jointly determined by boundary conditions and NNN hopping. Under open boundary conditions, non-Hermiticity drives the system to exhibit the skin effect, resulting in boundary-localized eigenstates characterized by an IPR that is insensitive to system size and remains high. The NNN hopping C_5 acts as a tuning parameter that competes with the skin effect, gradually suppressing boundary localization and driving the system toward an extended phase. Under PBCs, translational symmetry eliminates the boundary dependence of the skin effect, and eigenstates remain extended, with the IPR scaling to zero as L increases.

To quantitatively characterize the finite-size scaling of the IPR, we fit the OBC data in figure 14(a) with the asymptotic form

$$\text{IPR}(L) \sim \begin{cases} \text{const}, & C_5 \ll C_5^{(\text{skin})}, \\ L^{-\alpha}, & C_5 \gtrsim C_5^{(\text{skin})}, \end{cases} \quad (20)$$

where $C_5^{(\text{skin})} \approx 500 \text{ pF}$ marks the onset of NHSE suppression. For $C_5 = 0$, the IPR saturates to a finite value $\text{IPR} \approx 0.32$ as L increases, confirming that the localization length is much smaller than the system size. In stark contrast, at $C_5 = 500 \text{ pF}$, the IPR exhibits a power-law decay $\text{IPR} \sim L^{-\alpha}$ with $\alpha \approx 0.9$, extracted from the slope of the log-log plot. This value is close to the ideal extended-state limit $\alpha = 1$,

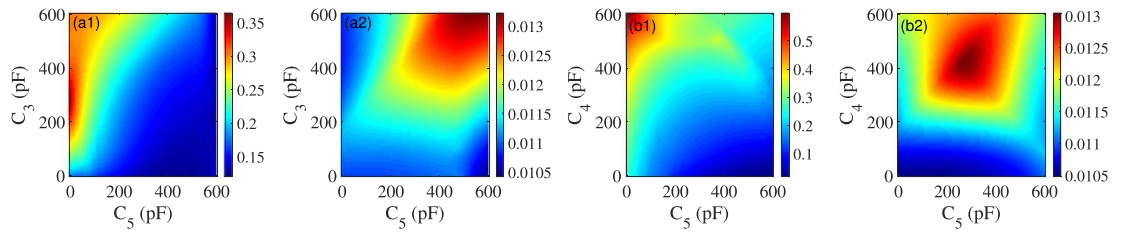


Figure 15. (a) For $C_4 = 200$ pF, IPR as a function of C_5 and C_3 under (a1) OBC and (a2) PBC. (b) For $C_3 = 100$ pF, IPR as a function of C_5 and C_4 under (b1) OBC and (b2) PBC. Other circuit parameters are as in figure 5.

indicating that the eigenstates have become fully extended in the thermodynamic limit. The gradual increase of α with C_5 (from 0 to 0.9) mirrors the progressive contraction of the GBZ toward the unit circle, providing a direct link between finite-size scaling and the non-Bloch geometric deformation discussed. Under PBC, the IPR follows $\text{IPR} \sim 1/L$ for all C_5 , consistent with the expected Bloch-wave scaling and further confirming the absence of the skin effect under periodic boundaries.

We have previously explored the competition between NNN hopping and non-hermite skin effect, but mainly through indirect methods. To more intuitively reveal the interaction between the two, we directly investigated their competitive relationship. Figure 15 further investigates the dependence of the IPR on the NNN hopping C_5 and another coupling parameter (C_3 or C_4) in the non-Hermitian SSH circuit model, under both OBC and PBC. Panel (a) fixes $C_4 = 200$ pF and examines the influence of C_5 and C_3 on the IPR, while panel (b) fixes $C_3 = 100$ pF and examines the influence of C_5 and C_4 on the IPR. As a quantitative measure of eigenstate localization, the IPR directly reflects the competition between the NHSE and different coupling channels. In figures 15(a1) and (a2) (with C_4 fixed), when C_3 is small, the IPR remains low and exhibits little sensitivity to variations in C_5 under both OBC and PBC. As C_3 increases, the IPR under OBC (figure 15(a1)) increases significantly and shows a further decreasing trend with increasing C_5 , suggesting that C_3 , as an additional coupling channel, competes with the NNN hopping C_5 to keep the boundary localization. Under PBC (figure 15(a2)), the IPR remains consistently low and shows little variation with C_5 and C_3 , once again confirming that eigenstates are extended under PBCs.

Similar behavior is observed in figures 15(b1) and (b2) (with C_3 fixed). When C_4 is small, the IPR under OBC (figure 15(b1)) is low, indicating skin effect is weak. As C_4 increases, the IPR gradually increases, but the enhancement of C_5 further promotes the suppression of localization. This demonstrates that C_4 also has a competitive relationship with NNN hopping. Under PBC (figure 15(b2)), the IPR remains low and exhibits little variation with the parameters, further validating the decisive role of boundary conditions in determining localization behavior. In the non-Hermitian SSH circuit model, the degree of eigenstate localization is jointly determined by multiple coupling parameters (C_3 , C_4 , and C_5) and the boundary conditions. Under open boundary conditions, the NHSE tends to localize eigenstates at the boundary, while the coupling channels C_3 , C_4 , and C_5 can individually influence the skin effect to modulate this localization. Furthermore, when multiple coupling effects are simultaneously enhanced, these coupling interactions exhibit distinctly different behaviors, thereby further modulating localization phenomena. Under PBCs, translational symmetry eliminates the foundation for boundary localization, and eigenstates remain extended, with the IPR staying at low values. These results not only deepen the understanding of multi-parameter competition mechanisms in non-Hermitian systems but also provide a clear parameter-tuning pathway for flexibly controlling the skin effect using circuit platforms.

The extensive analyses above ranging from the evolution of admittance and energy spectra to the spatial distributions of eigenstates, impedance profiles, and the inverse participation ratio consistently demonstrate that NNN hopping significantly suppresses the NHSE and drives the system from a boundary-localized phase toward a bulk-extended regime. However, while these observables capture the qualitative and quantitative changes in localization behavior, they do not directly address the underlying topological nature of the phase transition. A complete understanding of how NNN hopping influences the global topological properties requires a framework capable of restoring the bulk boundary correspondence in non-Hermitian systems. In the following section, we therefore turn to non-Bloch band theory, which introduces the generalized Brillouin zone and non-Bloch topological invariants, allowing us to rigorously characterize the topological phase transitions and reveal the role of NNN hopping in reshaping the system topological order. The analyses above ranging from spectral evolution and eigenstate distributions to impedance profiles and IPR-consistently demonstrate that NNN hopping suppresses the NHSE

and drives the system from a boundary-localized phase toward a bulk-extended regime. However, these observables, while quantitatively capturing localization behavior, do not directly address the underlying topological nature of the phase transition. A complete understanding requires a theoretical framework capable of restoring the bulk-boundary correspondence in non-Hermitian systems. In the following Section, we therefore turn to non-Bloch band theory, which introduces the GBZ and non-Bloch topological invariants, allowing us to rigorously characterize the topological phase transitions and reveal the role of NNN hopping in reshaping the system topological order.

5. Non-Bloch band theory and topological invariants

5.1. GBZ deformation

In contrast to Hermitian systems, non-Hermitian systems exhibit the distinctive NHSE, which leads to a fundamental disparity between wave functions under periodic and open boundary conditions. The most striking consequence is the dramatic mismatch between their energy spectra. This breakdown invalidates the conventional bulk-boundary correspondence for describing topological properties in such systems. To construct a self-consistent topological theory under open boundaries, it is essential to introduce the GBZ and non-Bloch topological invariants, which correctly capture the bulk-boundary correspondence in non-Hermitian settings. We begin with the one-dimensional non-Hermitian SSH model incorporating a NNN hopping term t_3 , whose real-space Hamiltonian is given by equation (1). Writing the eigenvector as $|\psi\rangle = (\dots, \psi_{1,A}, \psi_{1,B}, \psi_{2,A}, \psi_{2,B}, \dots)^T$ and inserting it into the Schrödinger equation $H|\psi\rangle = E|\psi\rangle$ yields the coupled difference equations

$$E\psi_{n,A} = \left(t_2 + \frac{\gamma_2}{2}\right)\psi_{n-1,B} + \left(t_1 - \frac{\gamma_1}{2}\right)\psi_{n,B} + t_3\psi_{n+1,B}, \quad (21)$$

$$E\psi_{n,B} = t_3\psi_{n-1,A} + \left(t_1 + \frac{\gamma_1}{2}\right)\psi_{n,A} + \left(t_2 - \frac{\gamma_2}{2}\right)\psi_{n+1,A}. \quad (22)$$

We seek solutions that are superpositions of exponential modes,

$$\psi_{n,\mu} = \sum_j \phi_{n,\mu}^{(j)}, \quad \mu = A, B, \quad (23)$$

with

$$\phi_{n,A}^{(j)} = \beta_j^n \phi_A^{(j)}, \quad \phi_{n,B}^{(j)} = \beta_j^n \phi_B^{(j)}. \quad (24)$$

Substituting this ansatz into equations (21) and (22) gives

$$\left[\left(t_2 + \frac{\gamma_2}{2}\right)\beta_j^{-1} + \left(t_1 - \frac{\gamma_1}{2}\right) + t_3\beta_j\right]\phi_B^{(j)} = E\phi_A^{(j)}, \quad (25)$$

$$\left[t_3\beta_j^{-1} + \left(t_1 + \frac{\gamma_1}{2}\right) + \left(t_2 - \frac{\gamma_2}{2}\right)\beta_j\right]\phi_A^{(j)} = E\phi_B^{(j)}. \quad (26)$$

These equations can be compactly written as a generalized Bloch Hamiltonian

$$\mathcal{H}(\beta) = R_+(\beta)\sigma_+ + R_-(\beta)\sigma_-, \quad (27)$$

where

$$R_+(\beta) = \left(t_2 + \frac{\gamma_2}{2}\right)\beta^{-1} + \left(t_1 - \frac{\gamma_1}{2}\right) + t_3\beta, \quad (28)$$

$$R_-(\beta) = t_3\beta^{-1} + \left(t_1 + \frac{\gamma_1}{2}\right) + \left(t_2 - \frac{\gamma_2}{2}\right)\beta, \quad (29)$$

and $\sigma_{\pm} = (\sigma_x \pm i\sigma_y)/2$. The eigenvalue equation

$$\det[\mathcal{H}(\beta) - E] = 0 \quad (30)$$

then becomes

$$E^2 = \left[\left(t_2 + \frac{\gamma_2}{2}\right)\beta^{-1} + \left(t_1 - \frac{\gamma_1}{2}\right) + t_3\beta\right] \times \left[t_3\beta^{-1} + \left(t_1 + \frac{\gamma_1}{2}\right) + \left(t_2 - \frac{\gamma_2}{2}\right)\beta\right]. \quad (31)$$

Equation (31) is an algebraic equation for β of even degree $2M$ in general. The condition for the formation of continuous bands can be written as

$$|\beta_M| = |\beta_{M+1}|, \quad (32)$$

where the solutions β_j are ordered such that $|\beta_1| \leq |\beta_2| \leq \dots \leq |\beta_{2M}|$. The locus of β_M and β_{M+1} satisfying this equality defines the GBZ C_β . In a Hermitian system one has $|\beta_M| = |\beta_{M+1}| = 1$, and C_β reduces to the unit circle $|\beta| = 1$. For the present one-dimensional non-Hermitian SSH model with NNN hopping, equation (31) is quartic in β , yielding four solutions β_i ($i = 1, 2, 3, 4$). Sorting them by magnitude gives $|\beta_1| \leq |\beta_2| \leq |\beta_3| \leq |\beta_4|$, and the continuum-band condition simplifies to

$$|\beta_2| = |\beta_3|. \quad (33)$$

To solve equation (31) under the constraint (33), we set $E^2 = f(\beta)$ and suppose that two solutions β and β' share the same modulus: $|\beta| = |\beta'|$. Writing β'^{θ} with real θ and imposing $E^2 = f(\beta e^{i\theta}) = 0$ leads to

$$\begin{aligned} 0 = & \left(t_2 + \frac{\gamma_2}{2}\right) t_3 \beta^{-2} (1 - e^{-2i\theta}) \\ & + \left[\left(t_1 + \frac{\gamma_1}{2}\right) \left(t_2 + \frac{\gamma_2}{2}\right) + \left(t_1 - \frac{\gamma_1}{2}\right) t_3\right] \beta^{-1} (1 - e^{-i\theta}) \\ & + \left[\left(t_1 + \frac{\gamma_1}{2}\right) t_3 + \left(t_1 - \frac{\gamma_1}{2}\right) \left(t_2 - \frac{\gamma_2}{2}\right)\right] \beta (1 - e^{i\theta}) \\ & + \left(t_2 - \frac{\gamma_2}{2}\right) t_3 \beta^2 (1 - e^{2i\theta}), \end{aligned} \quad (34)$$

with $\theta \in (0, 2\pi)$. Thus the problem of finding β is transformed into solving for θ . Once a permissible θ is obtained from equation (34), the corresponding β satisfying $|\beta| = |\beta e^{i\theta}|$ gives a point on the GBZ. Combining this with the ordering condition $|\beta_2| = |\beta_3|$, we identify β and β' with β_2 and β_3 , respectively. Consequently, the GBZ C_β is precisely the set of β values that fulfill $|\beta| = |\beta'|$. After deriving the theoretical formula for the GBZ, we further investigate the effect of NNN hopping terms on its structure.

The above GBZ construction provides a geometric foundation for distinguishing two distinct critical phenomena that occur at different parameter values in our model. The first is the suppression of the NHSE, which is quantitatively characterized by the contraction of the GBZ towards the unit circle. We define the radial deformation ratio

$$r_{\text{GBZ}} \equiv \frac{|\beta_4|}{|\beta_1|}, \quad (35)$$

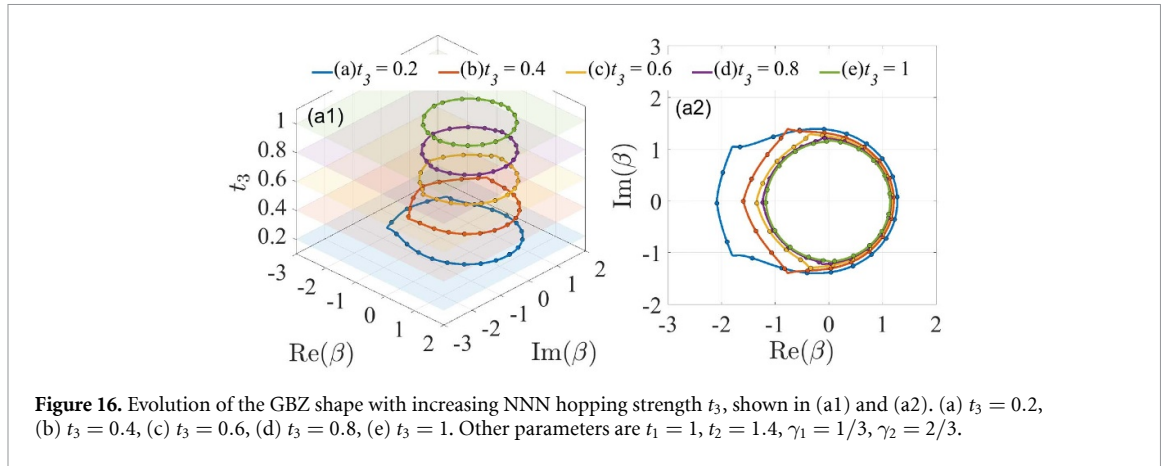
where β_1 and β_4 are respectively the smallest and largest moduli among the four solutions of equation (30). In a Hermitian limit $r_{\text{GBZ}} = 1$, whereas a strong NHSE corresponds to $r_{\text{GBZ}} \gg 1$. As the NNN hopping t_3 increases, r_{GBZ} monotonically decreases, and the condition

$$r_{\text{GBZ}} \rightarrow 1 \iff \text{NHSE fully suppressed}, \quad (36)$$

yields a critical value $t_3^{(\text{skin})}$ (or equivalently $C_5^{(\text{skin})}$). This is a metric condition on the spectral locus: it indicates when the open-boundary spectrum collapses onto the periodic-boundary one, irrespective of topological invariants. The second criticality is the topological phase transition, which is governed by the non-Bloch winding number ν . According to the argument principle (see appendix B), a change in ν occurs only when a zero of either $R_+(\beta)$ or $R_-(\beta)$ crosses the GBZ contour C_β . Mathematically, this happens when there exists a point $\beta_0 \in C_\beta$ such that

$$R_\pm(\beta_0) = 0, \quad \text{with} \quad |\beta_0| = |\beta'| \text{ for the paired GBZ solutions.} \quad (37)$$

Equivalently, the gap-closing condition $R_+(\beta)R_-(\beta) = 0$ on the GBZ defines the topological transition point $t_3^{(\text{topo})}$ (and the corresponding $C_5^{(\text{topo})}$). This is a homotopy condition: the winding number is a topological invariant that can only change when the integration contour is pierced by a zero of the off-diagonal entries. Importantly, $t_3^{(\text{skin})}$ and $t_3^{(\text{topo})}$ do not coincide in general. For our parameter set, $C_5^{(\text{skin})} \approx 500 \text{ pF}$ (where the OBC and PBC spectra nearly overlap), while $C_5^{(\text{topo})} \approx 590 \text{ pF}$ (where the non-Bloch winding number jumps). This separation stems from the fact that the radial contraction of the GBZ is governed by the moduli of β , whereas the zero-crossing condition depends on both the moduli and the phases of the roots. In geometric terms, the GBZ may approach the unit circle (suppressing



the skin effect) before any of the zeros of $R_{\pm}(\beta)$ actually cross the contour; the topological transition requires a specific trajectory of zeros through the GBZ. Hence, the two phenomena are controlled by different functions of the circuit parameters, and their non-coincidence is a hallmark of non-Bloch bulk-boundary correspondence in the presence of long-range couplings. This separation is explicitly demonstrated in our numerical results (figures 16 and 17), where the GBZ becomes nearly circular at $t_3 \simeq 0.9$ (corresponding to $C_5 \sim 500$ pF), while the winding number remains quantized until a larger t_3 value is reached. Only when the zeros cross the GBZ does the winding number jump, signaling the true topological phase transition.

We note that in practical circuit implementations, parasitic effects such as stray capacitances, parasitic inductances, and series resistances may quantitatively modify the GBZ. Specifically, these effects can: (i) alter the frequency dependence of the effective couplings, shifting the solutions β that define the GBZ contour; (ii) introduce unintended hopping channels between non-adjacent nodes through stray couplings; and (iii) break the precise symmetry of forward and backward hopping amplitudes via routing-induced crosstalk, potentially shifting the critical parameters of the band gap closure. Nevertheless, previous experimental studies on non-Hermitian topological circuits have demonstrated that the GBZ topology remains robust against moderate parasitic effects, and the qualitative behavior, namely the continuous deformation of the GBZ toward the unit circle with increasing NNN hopping, remains unchanged. A detailed quantitative analysis of these effects is left for future experimental work. Recent studies have shown that NHSE can significantly affect noise propagation, further motivating the distinction between deterministic eigenmode profiles and fluctuation-sensitive observables.

Figure 16 illustrates the evolution of the GBZ shape as the NNN hopping strength t_3 increases. When t_3 is relatively small, with t_3 ranging from 0.2 to 0.8, the GBZ exhibits a distinctly non-circular shape. This distorted geometry is a direct signature of the NHSE, where eigenstates acquire complex wavevectors and localize exponentially at the boundaries. As t_3 increases, the GBZ undergoes a systematic transformation toward a unit circle. The progression from figures 16(a1) to (a2) shows the curve contracting and gradually approaching circular geometry. This evolution corresponds to the weakening of NHSE due to enhanced inter-site connectivity provided by NNN hopping, which dilutes boundary effects and promotes more uniform bulk distribution of eigenstates. When t_3 reaches 1 (figure 16(a2)), the GBZ closely approximates a perfect unit circle. This indicates that the system eigenstates have largely recovered extended, Bloch-like character with significantly reduced sensitivity to boundary conditions. The observed convergence suggests that beyond a critical t_3 value, the GBZ would collapse entirely onto the unit circle, signaling a transition from non-Hermitian localized states to Hermitian-like extended states. This visual narrative demonstrates how NNN hopping parameter t_3 acts as a control knob, driving the system from strong non-Hermitian behavior with prominent boundary localization toward near-Hermitian regime with bulk-extended states, providing theoretical guidance for manipulating non-Hermitian topological properties through coupling range engineering. The continuous deformation of the GBZ toward the unit circle directly reflects the suppression of the NHSE and suggests a corresponding evolution in the system topological properties. To quantitatively characterize this evolution, we now examine the non-Bloch topological invariants defined on the GBZ contour. These invariants provide a rigorous framework for identifying topological phase transitions and establishing the bulk-boundary correspondence in non-Hermitian systems. From the mapping perspective, these parasitic effects modify the ideal correspondence $C_3 \leftrightarrow \gamma_1/2$, $C_4 \leftrightarrow \gamma_2/2$, and $C_5 \leftrightarrow t_3$ in several ways: (i) stray capacitances renormalize the coupling strengths as $C_i \rightarrow C_i + \delta C_i$; (ii) unintended hopping channels between non-adjacent

nodes introduce additional terms in $h_{AB}(k)$ and $h_{BA}(k)$; and (iii) routing-induced crosstalk breaks the precise symmetry between forward and backward hopping amplitudes. Nevertheless, these corrections are perturbative and do not close the bulk gap or alter the topological phase transition qualitatively in the parameter regime studied in this work, as demonstrated in previous non-Hermitian topological circuit experiments.

5.2. Evolution of non-Bloch topological invariants

We further validate our findings by examining the influence of NNN hopping on topological invariants. In non-Hermitian systems, the conventional bulk-boundary correspondence breaks down, and standard topological invariants fail to accurately capture the system properties. We therefore employ non-Bloch topological invariants for a faithful description. First, we can determine that the eigenvalue of the Hamiltonian is $E_{\pm}(\beta) = \pm\sqrt{R_+(\beta)R_-(\beta)}$ by applying the equation (27). For the eigenvectors $E_+(\beta)$ and $E_-(\beta)$, we obtain their normalized left and right eigenvectors, which can be expressed as

$$|u_{R,+}\rangle = \frac{1}{\sqrt{2}\sqrt{R_+R_-}} \begin{pmatrix} R_+ \\ \sqrt{R_+R_-} \end{pmatrix}, \quad (38)$$

$$|u_{R,-}\rangle = \frac{1}{\sqrt{2}\sqrt{R_+R_-}} \begin{pmatrix} R_+ \\ -\sqrt{R_+R_-} \end{pmatrix}, \quad (39)$$

$$\langle u_{L,+}| = \frac{1}{\sqrt{2}\sqrt{R_+R_-}} (R_-, \sqrt{R_+R_-}), \quad (40)$$

$$\langle u_{L,-}| = \frac{1}{\sqrt{2}\sqrt{R_+R_-}} (R_-, -\sqrt{R_+R_-}). \quad (41)$$

Next, we define the matrix $Q(\beta)$, which can be written as

$$\begin{aligned} Q(\beta) &= (|u_{R,+}\rangle\langle u_{L,+}| - |u_{R,-}\rangle\langle u_{L,-}|) = \begin{pmatrix} 0 & q(\beta) \\ q^{-1}(\beta) & 0 \end{pmatrix} \\ &= \frac{1}{\sqrt{R_+(\beta)R_-(\beta)}} \begin{pmatrix} 0 & R_+(\beta) \\ R_-(\beta) & 0 \end{pmatrix}, \end{aligned} \quad (42)$$

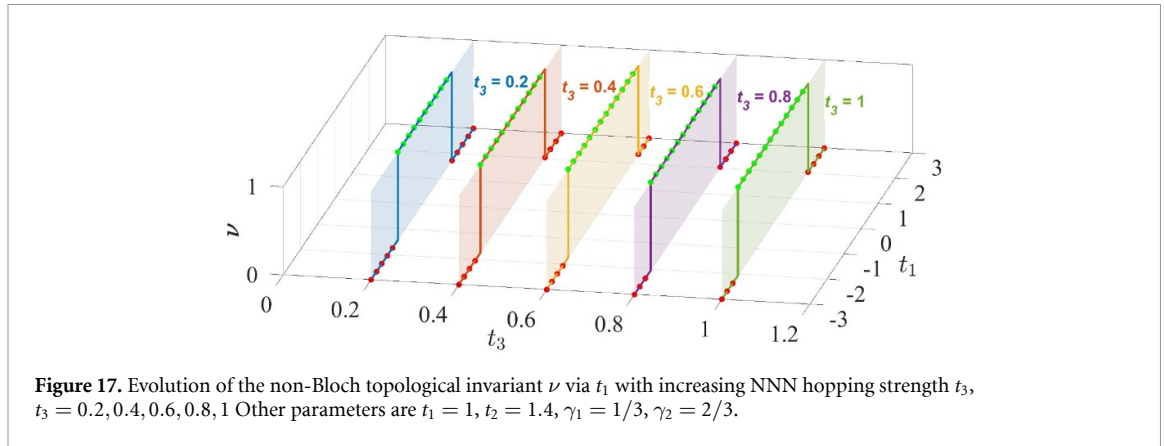
where

$$q(\beta) = \frac{R_+(\beta)}{\sqrt{R_+(\beta)R_-(\beta)}}, \quad q^{-1}(\beta) = \frac{R_-(\beta)}{\sqrt{R_+(\beta)R_-(\beta)}}, \quad (43)$$

Finally, we compute the winding number w for the Hamiltonian $H(\beta)$, defined as follows

$$\begin{aligned} \nu &= \frac{i}{2\pi} \int_{C_\beta} dq q^{-1}(\beta) \\ &= -\frac{1}{2\pi} \frac{[\arg R_+(\beta) - \arg R_-(\beta)]_{C_\beta}}{2} \\ &= -\frac{\nu_+ - \nu_-}{2}, \end{aligned} \quad (44)$$

where $\nu_{\pm} = \frac{1}{2\pi} [\arg R_{\pm}(\beta)]_{C_\beta}$, $[\arg R_{\pm}(\beta)]_{C_\beta}$ denotes the phase change of $R_{\pm}(\beta)$ as β traverses the contour C_β along the GBZ in the counterclockwise direction. The winding number takes the value $\nu = 1$ when the system is in a topological phase, and $\nu = 0$ in an ordinary phase. Topological phase transitions are consequently identified by changes in the non-Bloch winding number ν . We emphasize that the quantization of the non-Bloch winding number ν is preserved throughout the tuning process of the NNN hopping strength t_3 . This can be rigorously understood as follows. The winding number ν is defined on the GBZ contour C_β as ν , which counts the winding of the phase functions $R_{\pm}(\beta)$ as β traverses the closed contour. As t_3 varies, both the GBZ contour and the functions $R_{\pm}(\beta)$ deform continuously. As long as the point gap remains open i.e. $R_+(\beta)R_-(\beta) \neq 0$ on C_β , the integration path can deform continuously without crossing any singularities. According to the argument principle (equivalently, the homotopy invariance of winding numbers), the winding number remains quantized and cannot change during this continuous deformation. Only when the point gap closes at the critical coupling t_3^c do the zeros of $R_{\pm}(\beta)$ traverse the GBZ contour, causing the winding number to jump discretely from $\nu = 1$ to $\nu = 0$. This gap-closing condition corresponds precisely to the topological phase transition. Therefore, NNN hopping merely shifts the location of the phase transition (the critical parameter t_3^c)



without altering the quantized values of the topological invariant itself. A detailed analytical derivation is provided in appendix B.

Figure 17 illustrates the evolution of the non-Bloch topological invariant with increasing NNN hopping strength t_3 . Together with figure 16, which depicts the deformation of the GBZ, this figure provides a complete picture of how the topological properties of the system are controlled. As shown, with t_3 increasing stepwise from 0.2 to 1, the topological invariant exhibits a clear jump. This change is directly linked to the contraction of the GBZ from a non circular shape toward the unit circle in figure 16, indicating that NNN hopping not only alters the spatial distribution of eigenstates but also profoundly affects the topological order of the system. From a physical perspective, the jump in the topological invariant signals a non-Hermitian topological phase transition. When t_3 is small, the strong NHSE causes significant GBZ deformation away from the unit circle. At the same time, the non-Bloch winding number takes the value $\nu = 1$, indicating the presence of topologically protected boundary states. These two features? GBZ deformation and topological invariant?are distinct: the former signals the NHSE (localization of bulk states), while the latter determines the topological phase. As t_3 increases, the NHSE is gradually suppressed (the GBZ contracts toward the unit circle), and separately, the system undergoes a topological phase transition when the gap closes, causing the winding number to jump from $\nu = 1$ to $\nu = 0$.

More specifically, the topological invariant ν is defined on the GBZ contour C_β . When $t_3 < t_3^c$, the GBZ is a closed loop that encloses the origin, and the winding number $\nu = 1$ corresponds to a topologically nontrivial phase with boundary-localized skin modes. As t_3 increases and approaches t_3^c , the GBZ contracts and the band gap narrows. At $t_3 = t_3^c$, the band gap closes, and the GBZ passes through the origin, leading to a singularity in the winding number calculation. For $t_3 > t_3^c$, the GBZ no longer encloses the origin, and the winding number becomes $\nu = 0$, corresponding to a topologically trivial phase with extended bulk states. Thus, the jump of ν is a direct consequence of the band topology change driven by the continuous deformation of the GBZ and the associated band gap closure, establishing a clear and quantitative connection between the NNN hopping parameter t_3 and the topological phase transition. Through multi-faceted analyses of the admittance spectrum, eigenstate distribution, impedance response, and inverse participation ratio, we have demonstrated that the NNN hopping C_5 (corresponding to the coupling strength t_3) serves as an effective parameter for modulating the topological and localization properties of the non-Hermitian SSH circuit. By enhancing the connectivity between lattice sites, it drives the GBZ to contract toward the unit circle, thereby significantly suppressing the NHSE and inducing a continuous transition from boundary-localized to bulk-extended states. These results, grounded in observable spectral and spatial properties, provide a comprehensive foundation for understanding the role of NNN hopping in non-Hermitian topological systems. The above analysis establishes a quantitative connection between NNN hopping and the topological phase transition, where the discrete jump of the non-Bloch winding number reflects the continuous deformation of the GBZ and the associated band gap closure, providing a rigorous theoretical foundation beyond the intuitive picture of enhanced bulk connectivity. It is worth noting that the parameter value at which the OBC and PBC spectra nearly overlap ($C_5 \sim 500$ pF) does not coincide with the topological transition threshold t_3^c . This is because these two critical points are governed by distinct physical mechanisms: the former corresponds to the geometric condition of GBZ contraction toward the unit circle (suppression of the NHSE), while the latter corresponds to the topological condition of zero-crossing of the GBZ contour (change in

the non-Bloch winding number). These conditions are controlled by different functions of the parameters and are therefore not generally simultaneous, which is a hallmark of the boundary sensitivity inherent to non-Hermitian systems.

From an experimental perspective, the proposed circuit design can be readily implemented with standard components: the NNN hopping capacitance C_5 is realized by a conventional capacitor, while the non-Hermitian hopping terms (C_3 and C_4) can be implemented using INICs as schematically shown in figure 1(d), offering a clear pathway for future experimental verification. The finite-size scaling behavior warrants further discussion: near the critical point $C_5^c \approx 590$ pF, the localization length ξ diverges as $\xi \sim |C_5 - C_5^c|^{-1}$, and the NHSE is suppressed when $\xi \gtrsim L$, implying $|C_5 - C_5^c| \sim 1/L$ for larger arrays, while away from criticality ξ is finite and size-independent; our IPR analysis for different system sizes (figure 14) confirms that the qualitative trend persists in the thermodynamic limit with $1/L$ corrections to the critical point, consistent with previous circuit experiments in moderate-sized systems ($N \sim 10$ – 20 nodes). We further note that this mechanism may generalize to two-dimensional topological circuit networks, where NNN hoppings would connect sites in both directions and potentially give rise to richer phenomena such as corner states and higher-order topology; however, the critical parameter relationship would become more complex as the 2D GBZ is a region rather than a contour, and the critical parameters would likely depend on lattice geometry and boundary conditions, meaning a direct quantitative mapping from our 1D results may not hold universally, although the qualitative trend that increasing NNN hopping suppresses boundary localization is expected to remain valid. Beyond these theoretical extensions, we envision practical applications for a device capable of single-parameter tuning between boundary-localized and bulk-extended states: dynamically reconfigurable directional amplifiers with adjustable amplification direction and strength, sensors with tunable sensitivity exploiting the NHSE's extreme sensitivity to parameter variations, and more broadly, a reconfigurable circuit platform that can perform different functions amplification, sensing, or filtering simply by adjusting C_5 , offering a pathway toward integrated topological devices.

6. Conclusion

In summary, we have studied a one-dimensional non-Hermitian SSH model with NNN hopping on a circuit platform, focusing on deterministic linear-response properties. Our conclusions regarding the suppression of NHSE and topological phase transitions apply to admittance spectra, impedance profiles, and eigenmode distributions under coherent driving. They should not be interpreted as statements about physical correlations, noise distributions, or occupation probabilities, which would require a full Langevin or fluctuation analysis. We find that increasing the NNN hopping strength suppresses the NHSE, driving boundary-localized eigenstates into the bulk and reducing the inverse participation ratio. Under open boundary conditions, the impedance distribution transitions from a boundary peak to a uniform profile, while under PBCs, eigenstates remain extended. Using non-Bloch band theory, we show that NNN hopping continuously deforms the GBZ toward the unit circle and induces a discrete jump in the non-Bloch winding number, signaling a topological phase transition. Our results establish NNN hopping as an effective parameter for controlling non-Hermitian topological phases and provide a flexible circuit platform for realizing tunable topological devices in future experimental implementations. Importantly, beyond the intuitive expectation that bulk connectivity suppresses boundary localization, our work provides a quantitative framework connecting NNN hopping to GBZ deformation, non-Bloch topological invariants, and multi-parameter competition, thereby offering a comprehensive and experimentally accessible characterization of the NHSE suppression and the associated topological phase transition.

Acknowledgments

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Data availability statement

The data cannot be made publicly available upon publication because they are not available in a format that is sufficiently accessible or reusable by other researchers. The data that support the findings of this study are available upon reasonable request from the authors.

Appendix A. Impedance calculation of non-Hermit system

For a circuit with N nodes, the voltage and current at each node are $\mathbf{V} = (V_1, \dots, V_N)^T$ and $\mathbf{I} = (I_1, \dots, I_N)^T$. According to Kirchhoff's theorem, the two satisfy

$$\mathbf{I} = \mathbf{J}\mathbf{V}, \quad (\text{A1})$$

where $\mathbf{J}$ is the Laplace matrix of the circuit system (also called the admittance matrix). Under the condition of $\mathbf{J}$ being invertible, the inverse matrix is defined as the Green function G , which has the general form

$$G = J^{-1} = \begin{pmatrix} G_{11} & G_{12} & \cdots & G_{1N} \\ G_{21} & G_{22} & \cdots & G_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ G_{N1} & G_{N2} & \cdots & G_{NN} \end{pmatrix}, \quad (\text{A2})$$

where G_{ij} is the Green function coefficient between the nodes i and j , which is directly related to the impedance. The linear response relationship between voltage and current can be expressed as $\mathbf{V} = \mathbf{G}\mathbf{I}$. According to the above relation, the impedance between any two nodes A and B in the circuit can be solved. Assuming that an excitation current I is input to node A from the outside and then flows out from node B , all other nodes have zero current, then $I_A = I$, $I_B = -I$, $I_m = 0$. Substitute it into equation (A2)

$$\begin{pmatrix} V_A \\ V_B \end{pmatrix} = \begin{pmatrix} G_{AA} & G_{AB} \\ G_{BA} & G_{BB} \end{pmatrix} \begin{pmatrix} I \\ -I \end{pmatrix}, \quad (\text{A3})$$

According to Ohm's law, the impedance between two points is

$$Z_{AB} = \frac{V_A - V_B}{I} = G_{AA} + G_{BB} - G_{AB} - G_{BA}, \quad (\text{A4})$$

To further express the impedance in spectral form, we first examine the case where $\mathbf{J}$ is unitarily diagonalizable. Let $\mathbf{U}$ be a unitary matrix such that $\mathbf{U}^\dagger \mathbf{J} \mathbf{U} = \mathbf{j}$, where $\mathbf{j} = \text{diag}(j_1, \dots, j_N)$ is a diagonal matrix composed of eigenvalues. Since $G = J^{-1}$, the diagonalization yields $\mathbf{U}^\dagger \mathbf{G} \mathbf{U} = \mathbf{j}^{-1}$. Thus, the Green function G can be expressed as $G = \mathbf{U} \mathbf{j}^{-1} \mathbf{U}^\dagger$. Substituting the matrix elements yields

$$G_{AB} = \sum_{n=1}^N \frac{1}{j_n} U_{An} U_{Bn}^*. \quad (\text{A5})$$

Therefore, impedance can be expressed as

$$Z_{AB} = \sum_{n=1}^N \frac{1}{j_n} |U_{An} - U_{Bn}|^2. \quad (\text{A6})$$

If V_n denotes the normalized eigenvector of $\mathbf{J}$ with components V_{nA} and V_{nB} , the above expression can be equivalently written as

$$Z_{AB} = \sum_{n=1}^N \frac{1}{j_n} |V_{nA} - V_{nB}|^2. \quad (\text{A7})$$

When the circuit is non-Hermitian, the matrix $\mathbf{J}$ is generally not diagonalizable by unitary transformations, and it is necessary to distinguish between the right and left eigenstates. The right eigenstate $|V_n^r\rangle$ and the left eigenstate $\langle V_n^l|$ are defined as satisfying

$$\mathbf{J}|V_n^r\rangle = j_n|V_n^r\rangle, \quad \langle V_n^l|\mathbf{J} = j_n\langle V_n^l|. \quad (\text{A8})$$

Under the dual orthogonal normalization condition $\langle V_n^l | V_m^r \rangle = \delta_{nm}$, there exists a completeness relation

$$\sum_n |V_n^r\rangle \langle V_n^l| = I, \quad (\text{A9})$$

thus J^{-1} can be expanded in the spectrum

$$G = J^{-1} = \sum_n \frac{1}{j_n} |V_n^r\rangle \langle V_n^l|. \quad (\text{A10})$$

Therefore, the impedance between two points in a non-Hermitian system can be expressed as

$$Z_{AB} = \sum_{n=1}^N \frac{1}{j_n (V_n^l)^* V_n^r} (V_{nA}^l - V_{nB}^l) (V_{nA}^r - V_{nB}^r). \quad (\text{A11})$$

Appendix B. Analytical derivation of the quantization of the Non-Bloch winding number

In this appendix, we provide an analytical derivation of why the non-Bloch winding number remains quantized as the NNN coupling strength t_3 varies, with only the critical transition point being shifted. The definitions of the GBZ C_β and the non-Bloch winding number ν follow those given in section 5. The generalized Bloch Hamiltonian is $H(\beta) = R_+(\beta)\sigma_+ + R_-(\beta)\sigma_-$, with

$$R_+(\beta) = \left(t_2 + \frac{\gamma_2}{2}\right)\beta^{-1} + \left(t_1 - \frac{\gamma_1}{2}\right) + t_3\beta, \quad (\text{B1})$$

$$R_-(\beta) = t_3\beta^{-1} + \left(t_1 + \frac{\gamma_1}{2}\right) + \left(t_2 - \frac{\gamma_2}{2}\right)\beta. \quad (\text{B2})$$

The zeros of $R_+(\beta)$ satisfy $R_+(\beta) = 0$. Multiplying by β , we obtain a quadratic equation

$$t_3\beta^2 + \left(t_1 - \frac{\gamma_1}{2}\right)\beta + \left(t_2 + \frac{\gamma_2}{2}\right) = 0. \quad (\text{B3})$$

Thus the two zeros are

$$\beta_{\pm}^{(+)} = \frac{-(t_1 - \frac{\gamma_1}{2}) \pm \sqrt{(t_1 - \frac{\gamma_1}{2})^2 - 4t_3(t_2 + \frac{\gamma_2}{2})}}{2t_3}. \quad (\text{B4})$$

Similarly, the zeros of $R_-(\beta)$ are

$$\beta_{\pm}^{(-)} = \frac{-(t_1 + \frac{\gamma_1}{2}) \pm \sqrt{(t_1 + \frac{\gamma_1}{2})^2 - 4t_3(t_2 - \frac{\gamma_2}{2})}}{2(t_2 - \frac{\gamma_2}{2})}. \quad (\text{B5})$$

These expressions show explicitly that t_3 continuously deforms the positions of the complex zeros of $R_{\pm}(\beta)$. According to the argument principle, the winding number $\nu_{\pm} = \frac{1}{2\pi}[\arg R_{\pm}(\beta)]_{C_\beta}$ counts the difference between the number of zeros ($N_{\pm}$) and poles ($P_{\pm}$) of $R_{\pm}(\beta)$ enclosed by the GBZ contour C_β :

$$\nu_{\pm} = N_{\pm} - P_{\pm}. \quad (\text{B6})$$

Since both $R_+(\beta)$ and $R_-(\beta)$ have a simple pole at $\beta = 0$, and assuming that C_β encloses the origin, we have $P_{\pm} = 1$. Therefore,

$$\nu = \frac{\nu_- - \nu_+}{2} = \frac{N_- - N_+}{2}. \quad (\text{B7})$$

Thus, the non-Bloch winding number ν depends only on the numbers of zeros of $R_{\pm}(\beta)$ enclosed by the GBZ contour.

When t_3 is varied continuously, both the zeros of $R_{\pm}(\beta)$ and the GBZ contour C_β deform continuously. As long as no zero crosses C_β , the integers $N_{\pm}$ remain unchanged. Consequently, the winding number ν remains constant within a given gapped phase. This is precisely the homotopy invariance of the winding number: the map from C_β to the punctured complex plane $\mathbb{C} \setminus \{0\}$ cannot change its

homotopy class without crossing a singularity. A topological phase transition occurs only when a zero of $R_{\pm}(\beta)$ crosses the GBZ contour:

$$\exists \beta_c \in C_{\beta}, \quad R_{\pm}(\beta_c) = 0. \quad (\text{B8})$$

At this point, the gap closes ($R_+R_- = E^2 = 0$), and the winding number changes discontinuously. It is important to emphasize that the vanishing of the quadratic discriminant,

$$\left(t_1 - \frac{\gamma_1}{2}\right)^2 - 4t_3\left(t_2 + \frac{\gamma_2}{2}\right) = 0, \quad (\text{B9})$$

or

$$\left(t_1 + \frac{\gamma_1}{2}\right)^2 - 4t_3\left(t_2 - \frac{\gamma_2}{2}\right) = 0, \quad (\text{B10})$$

only indicates the coalescence of two zeros of the corresponding quadratic equations and is not sufficient to determine a topological phase transition. The actual transition condition requires that a zero lies exactly on the GBZ contour. In summary, t_3 continuously deforms the zeros of $R_{\pm}(\beta)$ and the GBZ contour C_{β} , but it does not change the integer value of the non-Bloch winding number within a given gapped phase. It merely shifts the phase boundaries (the critical parameter t_3^c) by modifying the condition under which a zero crosses the GBZ. The winding number changes only when the variation of t_3 drives a zero of $R_{\pm}(\beta)$ across C_{β} , corresponding precisely to the gap-closing condition of a topological phase transition. This analytical result confirms that the quantization of the non-Bloch topological invariant is preserved throughout the tuning process.

ORCID iDs

Cheng-Jun Han  0009-0002-6781-1118

Yi-Ping Wang  0000-0002-5323-2070

Ai-Xi Chen  0000-0002-7953-3752

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